II COMPITO 20.12.16 ESERCIZIO 1

1. $$\ln Q(V,T) = -2 \sum_{k} \ln\left(1 - e^{-\beta \hbar \omega_k}\right)$$

$$= -2 \int \frac{dk}{\frac{(2\pi)^3}{V}} \ln\left(1 - e^{-\beta \hbar c k}\right)$$

$$= \frac{V}{4\pi^3} 4\pi \int_0^\infty dk\, k^2 \ln\left(1 - e^{-\beta \hbar c k}\right)$$

$$= \frac{V}{\pi^2} \int_0^\infty dk\, k^2 \ln\left(1 - e^{-\beta \hbar c k}\right) \qquad s = \beta \hbar c k$$

$$\ln Q(V,T) = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty ds\, s^2 \ln\left(1 - e^{-s}\right)$$

$$= \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \sum_{n=1}^\infty \int_0^\infty ds\, s^2 \left(-\frac{e^{-ns}}{n}\right)$$

$$= -\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty dt\, t^2 e^{-t} = \frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \Gamma(3)\, \zeta(4)$$

$$\zeta(4) = \sum_{n=1}^\infty \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\ln Q(V,T) = -(V/\pi^2)(k_B T/\hbar c)^3 (\pi^4/45)$$

$$\boxed{\beta A(V,T) = -\ln Q(V,T) = -V\frac{\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3}$$

② $$S = -\frac{\partial A}{\partial T}\bigg|_V = -\frac{\partial}{\partial T}\left(-k_B T V \frac{\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3\right)$$

$$= k_B \frac{4\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3 V$$

$$\boxed{S = k_B \frac{4\pi^2}{45}\left(\frac{k_B T}{\hbar c}\right)^3 V}$$

③

$$P = -\frac{\partial A}{\partial V}\bigg|_T = -\frac{\partial}{\partial V}\left[-\frac{\pi^2}{45}\frac{(k_B T)^4}{(\hbar c)^3}V\right]$$

$$= \frac{\pi^2}{45}\frac{(k_B T)^4}{(\hbar c)^3}$$

$$P = \frac{\pi^2}{45}\frac{(k_B T)^4}{(\hbar c)^3}$$

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$$P = \frac{\pi^2}{45}\frac{(K_B T)^4}{(\hbar c)^3} = 1.01 \cdot 10^5 \text{ Pa}$$

$\boxed{S.I.}$ $(K_B T)^4 = 1.01 \cdot 10^5 (\hbar c)^3 \cdot 45 / \pi^2$

$$T = \left[1.01 \cdot 10^5 (\hbar c)^3 \cdot 45 / K_B^4 \pi^2\right]^{1/4}$$

$$= \left(1.01 \cdot 10^5 \cdot (1.06 \cdot 10^{-34} \cdot 3 \cdot 10^8)^3 \cdot 45 / \pi^2 / (1.38 \cdot 10^{-23})^4\right)^{1/4}$$

$$= \left(40.8 \cdot 10^5 \cdot 10^{-78} / 10^{-92}\right)^{1/4} = \left(40.8 \cdot 10^{19}\right)^{1/4}$$

$$= \left(4.08 \cdot 10^{20}\right)^{1/4} = 1.42 \cdot 10^5$$

$$\boxed{T = 1.42 \cdot 10^5 \text{ °K}}$$

ESERCIZIO 2.

① $$\rho_\sigma(H) = \int_{E<E_F} dE\, g_\sigma(E) = \frac{1}{2}\int_{E<E_F} dE\, g(E-\sigma\mu_B H) = \frac{1}{2}\int_{E<E_F} dE\, \frac{3\rho}{2E_F^{3/2}}\sqrt{E-\sigma\mu_B H}\,\Theta(E-\sigma\mu_B H)$$

$$= \frac{3\rho}{4E_F^{3/2}}\int_{\sigma\mu_B H}^{E_F} dE\,\sqrt{E-\sigma\mu_B H} = \frac{\rho}{2E_F^{3/2}}\left[E_F-\sigma\mu_B H\right]^{3/2}$$

$$\boxed{\rho_\sigma(H) = \frac{\rho}{2}\left[1-\frac{\sigma\mu_B H}{E_F}\right]^{3/2}}$$

② $$M(H,\rho) = -\frac{\mu_B\rho}{2}\left[\left(1-\frac{\mu_B H}{E_F}\right)^{3/2}-\left(1+\frac{\mu_B H}{E_F}\right)^{3/2}\right]$$

$$(1+y)^{3/2} = 1+\frac{3}{2}y+o(y^2)$$

$$M(H,\rho) \simeq -\frac{\mu_B\rho}{2}\left[1-\frac{3}{2}\frac{\mu_B H}{E_F} - 1 - \frac{3}{2}\frac{\mu_B H}{E_F}\right] = \frac{3}{2}\frac{\mu_B^2\rho}{E_F}H$$

$$\boxed{M(H,\rho) = \frac{3\mu_B^2\rho}{2E_F}H}$$

③ $$\chi_P = \left.\frac{\partial M}{\partial H}\right|_{H=0,\rho} = \frac{3\rho}{2E_F}\mu_B^2$$

$$\boxed{\chi_P = \frac{3\rho}{2E_F}\mu_B^2}$$

④ $$\boxed{\chi_P = \frac{3}{2}\frac{\rho}{E_F}\mu_B^2 = g(E_F)\mu_B^2}$$