COMPITO II

ESERCIZIO 1

① $$U = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} K_m \left[u(n) - u(n+m)\right]^2$$

② $$F_\ell = -\frac{\partial U}{\partial u(\ell)} = -\sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} K_m \left(u(n) - u(n+m)\right)\left[\delta_{n,\ell} - \delta_{n+m,\ell}\right]$$

$$= -\sum_{m=1}^{\infty} K_m \left[2u(\ell) - u(\ell+m) - u(\ell-m)\right]$$

③ Try with $u(\ell) = e^{i(q\ell a - \omega t)}$. This yields

$$M\omega^2 e^{i(q\ell a - \omega t)} = \sum_{m=1}^{\infty} K_m \left[2 - e^{iqma} - e^{-iqma}\right] e^{i(q\ell a - \omega t)}$$

$$\Rightarrow \quad M\omega^2 = 4 \sum_{m=1}^{\infty} K_m \sin^2\left(\frac{qa}{2}\right)$$

④ $$S = \sum_{m=1}^{\infty} K_m \left[2 - e^{iqma} - e^{-iqma}\right] = \sum_{m=1}^{\infty} K e^{-\mu m} \left[2 - e^{iqma} - e^{iqma}\right]$$

$$= K\left[2\left(\frac{1}{1-e^{-\mu}} - 1\right) - \left(\frac{1}{1-e^{-\mu+iqa}} - 1\right) - \left(\frac{1}{1-e^{-\mu-iqa}} - 1\right)\right]$$

$$= K\left[\frac{2}{1-e^{-\mu}} - \frac{2 - 2e^{-\mu}\cos(qa)}{\left(1-e^{-\mu}\cos(qa)\right)^2 + e^{-2\mu}\sin^2 qa}\right]$$

$$S = 2K\left[\frac{1}{1-e^{-\mu}} - \frac{1-e^{-\mu}\cos(qa)}{1+e^{-2\mu}-2e^{-\mu}\cos(qa)}\right]$$

$$= 2K\,\frac{\cancel{1}+e^{-2\mu}-2e^{-\mu}\cos(qa) - \cancel{1}+e^{-\mu}+e^{-\mu}\cos(qa) - e^{-2\mu}\cos(qa)}{(1-e^{-\mu})\left(1+e^{-2\mu}-2e^{-\mu}\cos(qa)\right)}$$

$$= \frac{2K}{1-e^{-\mu}}\,\frac{\left(e^{-\mu}+e^{-2\mu}\right)\left(1-\cos(qa)\right)}{1+e^{-2\mu}-2e^{-\mu}+2e^{-\mu}\left(1-\cos(qa)\right)}$$

$$= \frac{4K}{1-e^{-\mu}}\,\frac{\left(e^{-\mu}+e^{-2\mu}\right)\sin^2(qa/2)}{(1-e^{-\mu})^2+4e^{-\mu}\sin^2\left(\frac{qa}{2}\right)}$$

$$\omega^2 = \frac{4K}{M}\,\frac{1}{1-e^{-\mu}}\,\frac{\left(e^{-\mu}+e^{-2\mu}\right)\sin^2(qa/2)}{(1-e^{-\mu})^2+4e^{-\mu}\sin^2\left(\frac{qa}{2}\right)}$$

⑤ For $q \to 0$, to leading order

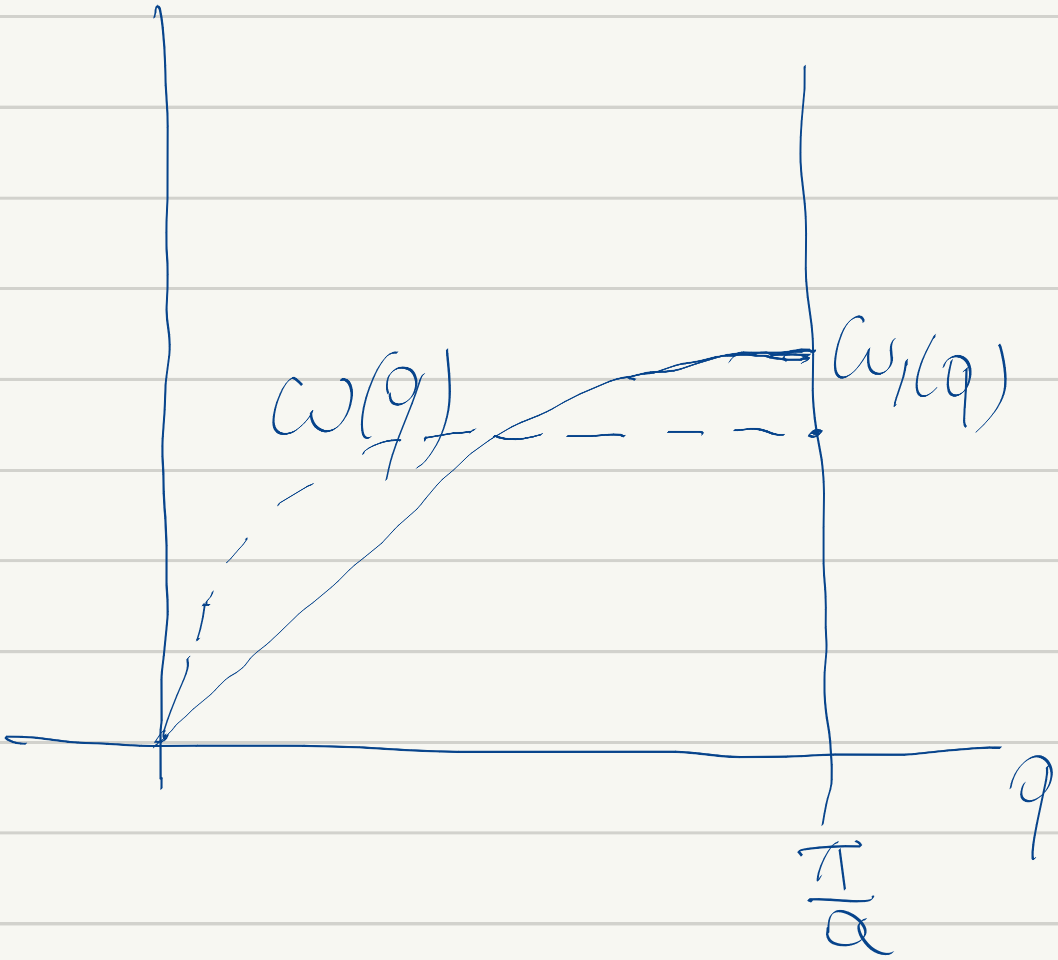
$$\omega^2 \simeq \frac{4K}{M}\,\frac{1}{(1-e^{-\mu})}\,\frac{\left(e^{-\mu}+e^{-2\mu}\right)}{(1-e^{-\mu})^2}\,\frac{q^2a^2}{4}$$

$$c_s = a\sqrt{\frac{K}{M}\,\frac{\left(e^{-\mu}+e^{-2\mu}\right)}{(1-e^{-\mu})^3}}$$

⑥ $$\omega(q) = \omega_1(q)\sqrt{\frac{\left(e^{-\mu}+e^{-2\mu}\right)}{(1-e^{-\mu})\left[(1-e^{-\mu})^2+4e^{-\mu}\sin^2\left(\frac{qa}{2}\right)\right]}}\,, \quad e^{-\mu}=\frac{1}{2}$$

$$\Rightarrow \quad \omega(q) = \omega_1(q)\sqrt{\frac{3/4}{\frac{1}{2}\left[\frac{1}{4}+4\cdot\frac{1}{2}\sin^2(qa/2)\right]}}$$

$$= \omega_1(q)\sqrt{\frac{6}{1+8\sin^2(qa)/2}}$$



$$C_s = \sqrt{6}\, c_{s,1}$$

$$\omega\left(\frac{\pi}{a}\right) = \omega_1\left(\frac{\pi}{a}\right)\sqrt{\frac{6}{9}}$$

In addition to the behavior around the origin and at $q = \pi/a$, inspecting $\frac{d\,\omega^2(q)}{dq}$ one discovers that it is greater than 0 for $0 < q < \frac{\pi}{a}$, thus as $\omega(q)$ is non negative it follows the $d\omega/dq > 0$ for $0 < q < \frac{\pi}{a}$

In detail,

$$\frac{M}{4K}\omega^2(q) \equiv \Omega^2(q) = \frac{6\sin^2(qa/2)}{1+8\sin^2 qa/2}$$

$$X = \sin^2(qa/2)$$

$$\frac{\partial}{\partial q}\Omega^2(q) = \frac{2X(1+8X)-16X^3}{(1+8X)^2}\cdot 6\frac{dX}{dq} = \frac{2X}{(1+8X)^2}6\frac{dX}{dq}$$

$$\frac{dX(q)}{dq} = 2\sin\left(\frac{qa}{2}\right)\cos\frac{qa}{2}\,\frac{a}{2} = \frac{a}{2}\sin(qa)$$

Hence, $\frac{\partial}{\partial q}\Omega^2(q) > 0 \qquad 0 < q < \frac{\pi}{a}$

① $C_v = \gamma T$ $\qquad \gamma = \frac{\pi^2}{3} k_B^2 g(\epsilon_F)$

$$g(\epsilon_F) = \frac{3}{\pi^2}\frac{\gamma}{k_B^2} = \frac{3 \times 1.85 \times 10^3}{\pi^2 (1.38 \times 10^{-16})^2}$$

$$= 2.95 \times 10^{34}\ \mathrm{erg}^{-1}\mathrm{cm}^{-3}$$

$$= 4.72 \times 10^{22}\ \mathrm{eV}^{-1}\mathrm{cm}^{-3}$$

② $\hbar\omega_D = k_B \Theta = 1.38 \times 10^{-16} \times 275$

$$= 3.79 \times 10^{-14}\ \mathrm{erg} = 2.37 \times 10^{-2}\ \mathrm{eV}$$

$$\omega_D = \frac{3.79 \times 10^{-14}}{1.05 \times 10^{-27}} = 3.61 \times 10^{13}\ \mathrm{s}^{-1}$$

③

$$n_0 = \frac{g(\epsilon_F)}{2} \qquad k_B T_c = 1.13\,\hbar\omega_D\, e^{-1/n_0 V_0}$$

$$(n_0 V_0)^{-1} = -\ln \frac{k_B T_c}{1.13\,\hbar\omega_D}$$

$$V_0 = \frac{2}{g(\epsilon_F)} \frac{1}{\ln \frac{1.13\,\hbar\omega_D}{k_B T_c}} =$$

$$= \frac{2}{4.72 \times 10^{22}} \frac{1}{\ln\left(\frac{1.13 \times 3.79 \times 10^{-14}}{1.38 \times 10^{-16}\ 9.26}\right)}$$

$$= 1.21 \times 10^{-23}\ \mathrm{eV\,cm}^3$$

④ $\Delta(0) = 1.76\, k_B T_c = 1.76 \times 8.62 \times 10^{-5} \times 9.26$

$= 1.41 \times 10^{-3}\ eV \qquad = 2.26 \times 10^{-15}\ erg$

⑤ $\xi_0 \simeq \dfrac{\epsilon_F}{k_F \Delta} = \dfrac{1}{\Delta} \dfrac{\hbar^2 k_F}{2m^*} = \dfrac{1}{\Delta} \dfrac{\hbar^2}{2m^*} (3\pi^2 n)^{\frac{1}{3}}$

$g(\epsilon_F) = \dfrac{3}{2} \dfrac{n}{\epsilon_F} = \dfrac{3}{2} \dfrac{n}{\hbar^2 k_F^2} 2m^* = \dfrac{8m^*}{\hbar^2} \dfrac{k_F}{8\pi^2}$

$k_F = \dfrac{8\pi^2 \hbar^2}{8m^*} g(\epsilon_F) = \dfrac{\pi^2 \times (1.05 \times 10^{-27})^2}{12 \times 9.11 \times 10^{-28}} \times 2.95 \times 10^{34}$

$= 2.94 \times 10^{7}\ cm^{-1}$

$\xi_0 = \dfrac{(1.05 \times 10^{-27})^2 \times 2.94 \times 10^{7}}{2.26 \times 10^{-15} \cdot 2 \cdot 12 \times 9.11 \times 10^{-28}}$

$= 6.18 \times 10^{-7}\ cm = 61.8\ \text{Å}$

⑥ $\Lambda = \left(\dfrac{m^* c^2}{4\pi n e^2}\right)^{\frac{1}{2}} = \left(\dfrac{12 \times 9.11 \times 10^{-28} \times 9 \times 10^{20}}{4\pi \times 8.58 \times 10^{20} \times (4.8)^2 \times 10^{-20}}\right)^{\frac{1}{2}}$

$= 6.29 \times 10^{-5}\ cm = 6290\ \text{Å}$

$n = \dfrac{k_F^3}{3\pi^2} = \dfrac{(2.94 \times 10^{7})^3}{3\pi^2} = 8.58 \times 10^{20}\ cm^{-3}$