Exercise 1 7/06/10

① $$U = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \sum_{m=1}^{\infty} C_m [u(n) - u(n+m)]^2$$

② $$M \ddot{u}(l) = -\frac{\partial}{\partial u(l)} U = -\sum_{m=1}^{\infty} C_m [2u(m) - u(n+m) - u(n-m)]$$

③ $$u(l) \propto e^{i(qal - \omega t)}$$

$$-M\omega^2 = -\sum_{m=1}^{\infty} C_m [2 - e^{iqam} - e^{-iqam}]$$

$$M\omega^2 = 4 \sum_{m=1}^{\infty} C_m \sin^2\left[\frac{qam}{2}\right]$$

④ $q \to 0$ $$M\omega^2 \simeq 4 \sum_{m=1}^{\infty} C_m m^2 \frac{q^2a^2}{4} = q^2a^2 \sum_{m=1}^{\infty} m^2 C_m$$

$$\Rightarrow \sum_{m=1}^{\infty} m^2 C_m < \infty \qquad c = a\sqrt{\sum_{m=1}^{\infty} m^2 C_m}$$

⑤ $$C_m = C_1 / m^p \qquad 1 < p < 3$$

$$c = a\sqrt{\sum_{m=1}^{\infty} C_1 m^{2-p}} \qquad -1 < 2-p < 1$$

$$\Rightarrow c = \infty$$

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$$\omega = 2\sqrt{\frac{C}{M}}\sqrt{\sum_{m=1}^{q}\frac{1}{m^p}\sin^2\frac{aqm}{2}}$$

$$= 2\sqrt{\frac{C}{M}}\cdot q\cdot X$$

$$X^2 = \sum_{m=1}^{\infty}\frac{1}{q^2}\frac{1}{m^p}\sin^2\frac{aqm}{2}$$

$$\simeq \int_1^{\infty} dm\,\frac{1}{m^p q^2}\sin^2\frac{aqm}{2} = \frac{1}{q^2}\left(\frac{aq}{2}\right)^{p-1}\cdot$$

$$\int_{q\frac{a}{2}}^{\infty} dy\,\frac{1}{y^p}\sin^2 y$$

$$A = \int_0^{\infty} dy\,\frac{1}{y^p}\sin^2 y < \infty \qquad 3 > p > 1$$

$$X^2 = A\left(\frac{a}{2}\right)^{p-1} q^{p-3} \qquad X = \sqrt{A\left(\frac{a}{2}\right)^{p-1}}\; q^{\frac{p-3}{2}}$$

$$\omega = 2\sqrt{\frac{C}{M}}\sqrt{A\left(\frac{a}{2}\right)^{p-1}}\; q^{\frac{p-1}{2}}$$

Exercise 2 7/06/10

① $g(\epsilon) = \frac{n}{\epsilon_F} \qquad \epsilon \geq 0$

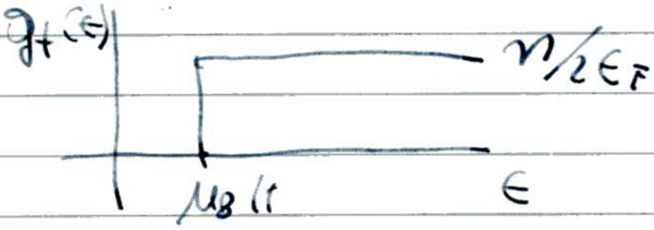
②
$$-(-\mu_B \cdot 2 \cdot S_z)H \quad \begin{cases} \mu_B H & S_z = \frac{1}{2} & \sigma = 1, + \\ -\mu_B H & S_z = -\frac{1}{2} & \sigma = -1, - \end{cases}$$

③
$$g_\sigma(\epsilon) = \frac{1}{A}\sum_{\vec{k}} \delta\left(\epsilon - \frac{\hbar^2 k^2}{2m} - \mu_B H \sigma\right)$$

$$= \frac{1}{A}\sum_{k} \delta\left(\epsilon_\sigma - \frac{\hbar^2 k^2}{2m}\right)$$

$$= \frac{1}{2} g(\epsilon_\sigma) \qquad \epsilon_\sigma = \epsilon - \mu_B H \sigma$$

$$g_\pm(\epsilon) = \frac{1}{2} g(\epsilon \mp \mu_B H) \quad \begin{cases} \epsilon > \mu_B H & \sigma = 1 \\ \epsilon > -\mu_B H & \sigma = -1 \end{cases}$$



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$$M = -\mu_B [n_+ - n_-]$$

$$= -\mu_B \left\{ \int_{\mu_B H}^{\infty} d\epsilon\, g_+(\epsilon) f(\epsilon) - \int_{-\mu_B H}^{\infty} d\epsilon\, g_-(\epsilon) f(\epsilon) \right\}$$

$$M = + \frac{\mu_B n}{2\epsilon_F} \left[\int_{-\mu_B H}^{\infty} d\epsilon\, f(\epsilon) - \int_{\mu_B H}^{\infty} d\epsilon\, f(\epsilon) \right]$$

$$M = \frac{\mu_B n}{2\epsilon_F} \int_{-\mu_B H}^{\mu_B H} d\epsilon\, f(\epsilon) = \frac{\mu_B n}{2\epsilon_F} \int_{-\mu_B H}^{\mu_B H} d\epsilon\, \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

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$$k_B T \ll \epsilon_F \quad \mu_B H \ll \epsilon_F \Rightarrow \mu \simeq \epsilon_F$$

$$\int_{-\mu_B H}^{\mu_B H} d\epsilon\, \frac{1}{e^{\beta(\epsilon-\mu)}+1} \simeq \int_{-\mu_B H}^{\mu_B H} d\epsilon\, \frac{1}{e^{\beta(\epsilon-\epsilon_F)}+1}$$

$$\simeq \int_{-\mu_B H}^{\mu_B H} d\epsilon = 2\mu_B H \Rightarrow M = \frac{n}{\epsilon_F} \mu_B^2 H$$

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$\chi = \mu_B^2 \frac{n}{\epsilon_F} = \mu_B\, g(\epsilon_F)$ identical to the 3-d result.

Additional info for exercise 2!

Note that for arbitrary T and H

$$M = \frac{\mu_B n}{2\epsilon_F} \int_{-\mu_B H}^{\mu_B H} d\epsilon \, \frac{1}{e^{\beta(\epsilon-\mu)}+1}$$

$$= \frac{\mu_B n}{2\epsilon_F} \left\{ -k_B T \ln\left[1+e^{-\beta(\epsilon-\mu)}\right] \right\} \Big|_{-\mu_B H}^{\mu_B H}$$

$$M = \frac{\mu_B n}{2\beta\epsilon_F} \ln\left[\frac{1+e^{\beta(\mu_B H+\mu)}}{1+e^{\beta(-\mu_B H-\mu)}}\right]$$

Also from

$$n = n_+ + n_- = \int_{\mu_B H}^{\infty} d\epsilon \, g_+(\epsilon) f(\epsilon) + \int_{-\mu_B H}^{\infty} d\epsilon \, g(\epsilon) f(\epsilon)$$

with a little algebra one gets

$$\beta\mu = \beta\epsilon_F + \ln\left[\sqrt{1+(\cosh^2\beta\mu_B H - 1)e^{-2\beta\epsilon_F}} - (\cosh\beta\mu_B H)e^{-\beta\epsilon_F}\right]$$