Esercizio 1 — 10 giugno 2010

①

$$E = N_1 \epsilon \Rightarrow N_1 = E/\epsilon \,,\quad N_0 = N - N_1 = N - E/\epsilon$$

Il numero di configurazioni con energia E è

$$\Omega = N!/[N_1!(N-N_1)!] = N!/[(\tfrac{E}{\epsilon})!\,(N-\tfrac{E}{\epsilon})!]$$

$$\simeq \left(\frac{N}{e}\right)^N \frac{1}{\left(\frac{N_1}{e}\right)^{N_1}\left(\frac{N-N_1}{e}\right)^{N-N_1}} = \frac{N^N}{N_1^{N_1}(N-N_1)^{N-N_1}}$$

②

$$\frac{S}{k_B} = N\ln N - N_1 \ln N_1 - (N-N_1)\ln(N-N_1)$$

$$= N\ln N - \frac{E}{\epsilon}\ln\frac{E}{\epsilon} - \left(N - \frac{E}{\epsilon}\right)\ln\left(N - \frac{E}{\epsilon}\right)$$

$$\frac{1}{k_B T} = \left.\frac{\partial S}{\partial k_B E}\right|_N = -\frac{1}{\epsilon}\ln\frac{E}{\epsilon} - \frac{1}{\epsilon} + \frac{1}{\epsilon}\ln\left(N - \frac{E}{\epsilon}\right) + \frac{1}{\epsilon}$$

$$\boxed{\frac{1}{k_B T} = \frac{1}{\epsilon}\ln\left(\frac{N\epsilon}{E} - 1\right)}$$

③

$$\frac{N\epsilon}{E} = 1 + e^{\beta\epsilon} \qquad \boxed{E = \frac{N\epsilon}{1+e^{\beta\epsilon}}}$$

(4) $N_1 = \frac{E}{\epsilon} = \frac{N}{1+e^{\beta\epsilon}}$

$N_0 = N - N_1 = N\left[1 - \frac{1}{e^{\beta\epsilon}+1}\right] = \frac{N e^{\beta\epsilon}}{1+e^{\beta\epsilon}}$

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1) $g(\epsilon) = \frac{1}{L} \sum_{\vec{p}}{}' \delta(E - \alpha|p|)$ P.B.C.

1D $\vec{p} \Rightarrow p$

$$g(\epsilon) = \frac{1}{L} \frac{L}{2\pi\hbar} \int_{-\infty}^{+\infty} dp \; \delta(E - \alpha|p|)$$

$\Rightarrow |p| = E/\alpha \qquad p = \pm E/\alpha$

$$g(\epsilon) = \frac{1}{h} \frac{1}{\alpha} \times 2 \, \vartheta(\epsilon)$$

$$\boxed{g(\epsilon) = \frac{2}{\alpha h} \vartheta(\epsilon)}$$

2)

$$\rho = \int d\epsilon \frac{g(\epsilon)}{\frac{1}{z} e^{\beta\epsilon} - 1} = \frac{2}{\alpha h} \int_0^\infty d\epsilon \frac{z e^{-\beta\epsilon}}{1 - z e^{-\beta\epsilon}}$$

$$= \frac{2}{\alpha h} (-) \beta^{-1} \int_z^0 \frac{d(z e^{-\beta\epsilon})}{1 - z e^{-\beta\epsilon}} \qquad \boxed{y = z e^{-\beta\epsilon}}$$

$$= \frac{2 k_B T}{\alpha h} \int_0^z \frac{dy}{1-y} = \frac{2 k_B T}{\alpha h} \left(-\ln(1-y)\right)\Big|_0^z$$

$$\boxed{\rho = -\frac{2}{\ell} \ln(1-z)}$$

(3) $$\beta \mathcal{P} L = \ln \mathcal{Z} = -\sum_p \ln\left(1 - z e^{-\beta\alpha|p|}\right)$$

$$= -L \int d\epsilon\, g(\epsilon) \ln\left(1 - z e^{-\beta\epsilon}\right)$$

$$\beta \mathcal{P} = -\frac{2}{\alpha h}\int_0^\infty d\epsilon \ln\left(1 - z e^{-\beta\epsilon}\right)$$

$$= -\frac{2 k_B T}{\alpha h}\int_0^\infty dt \ln\left(1 - z e^{-t}\right)$$

$$\boxed{\beta \mathcal{P} = \frac{2}{\ell} g_2(z)}$$

$$-\int_0^\infty dt \ln\left(1 - z e^{-t}\right) = +\sum_{n=1}^\infty \int_0^\infty dt \frac{\left(z e^{-t}\right)^n}{n}$$

$$= \sum_{n=1}^\infty \frac{z^n}{n}\int_0^\infty dt\, e^{-nt} = \sum_{n=1}^\infty \frac{z^n}{n^2}$$

(4) $$\rho = -\frac{2}{\ell} \ln 1 - z \Rightarrow z = 1 - e^{-\ell\rho/2}$$

$$z = \frac{\ell\rho}{2} - \left(\frac{\ell\rho}{2}\right)^2 \frac{1}{2} + \ldots$$

$$\beta \mathcal{P} = \frac{2}{\ell}\left[z + \frac{z^2}{4} + \ldots\right] = \frac{2}{\ell}\left[\frac{\ell\rho}{2} - \frac{\ell^2\rho^2}{8} + \frac{1}{4}\left(\frac{\ell\rho}{2} + \ldots\right)^2 + \ldots\right]$$

$$= \frac{2}{\ell}\left[\frac{\ell\rho}{2} - \frac{\ell^2\rho^2}{16} + \ldots\right]$$

$$\beta \mathcal{P} = \rho\left[1 - \frac{\ell\rho}{8} + \ldots\right]$$