ESERCIZIO 1 22/12/10

① $\mathcal{H} = \sum_{i=1}^{N} h_i \qquad h = \frac{p^2}{2m} + U \ln\left[\left(\frac{\sqrt{4\pi}\, r}{R}\right)^\alpha\right]$

$r \leq R$

$$Q = \frac{1}{N!}\left[\int \frac{d\bar{p}}{h^3} e^{-\frac{\beta p^2}{2m}} \cdot \int d\bar{r}\, e^{-\beta U(r)}\right]^N$$

$$\int \frac{d\bar{p}}{h^3} e^{-\frac{\beta p^2}{2m}} = \frac{1}{\lambda^3} \qquad \lambda^2 = \frac{h^2}{2\pi m k_B T}$$

$$\int d\bar{r}\, e^{-\beta U(r)} = 4\pi \int_0^R dr\, r^2\, e^{-\beta U \alpha \ln\left[\sqrt{4\pi}\frac{r}{R}\right]}$$

$$= 4\pi \int_0^R dr\, r^2\, e^{-\ln\left[\left(\sqrt{4\pi}\frac{r}{R}\right)^2\right]} \qquad \beta U \alpha = 2$$

$$= \cancel{4\pi} \int_0^R dr\, \cancel{r^2}\, \frac{R^2}{\cancel{r^2}} \frac{1}{\cancel{4\pi}} = R^2 \cdot R = R^3$$

$$Q = \frac{1}{N!}\left[\frac{R^3}{\lambda^3}\right]^N \simeq \left[\frac{e}{N}\left(\frac{R}{\lambda}\right)^3\right]^N$$

$$\boxed{A = -N k_B T \ln\left[\frac{e}{N}\frac{R^3}{\lambda^3}\right]}$$

(2) $$P = -\frac{\partial A}{\partial V} = -\frac{\partial R}{\partial V}\frac{\partial A}{\partial R} = -\frac{1}{\frac{\partial V}{\partial R}}\frac{\partial A}{\partial R}$$

$$V = \frac{4\pi}{3}R^3 \qquad \frac{\partial V}{\partial R} = 4\pi R^2$$

$$-\frac{\partial A}{\partial R} = N k_B T \frac{\partial}{\partial R} 3 \ln R = \frac{3 N k_B T}{R}$$

$$\boxed{P = \frac{1}{4\pi R^2}\frac{3 N k_B T}{R} = \frac{N}{V} k_B T}$$

(3) $$\mu = \frac{\partial A}{\partial N} = k_B T \ln \frac{N}{R^3}\frac{\lambda^3}{e} + k_B T$$

$$\boxed{\mu = k_B T \ln \frac{N}{R^3}\lambda^3 = k_B T \ln \frac{4\pi}{3}\frac{N}{V}\lambda^3}$$

(4) $$\rho(r) = N \langle \delta(\bar{r}-\bar{r}_1)\rangle = N \frac{\int d\bar{r}_1\, e^{-\beta U(\bar{r}_1)}\,\delta(\bar{r}-\bar{r}_1)}{\int d\bar{r}_1\, e^{-\beta U(\bar{r}_1)}}$$

$$= N \frac{e^{-\beta U(r)}}{R^3} = \frac{N}{R^3}\cdot\frac{R^2}{r^2}\frac{1}{4\pi} = \frac{1}{3}\frac{N}{V}\left(\frac{R}{r}\right)^2$$

$$\boxed{\rho(r) = \frac{1}{3}\frac{N}{V}\left(\frac{R}{r}\right)^2} \qquad \boxed{P = 3 k_B T \rho(R)} \qquad \boxed{\mu = k_B T \ln\left[4\pi\lambda^3\rho(R)\right]}$$

ESERCIZIO 2 22/12/10

① Le energie sono date da

$$E_{\{n_{\vec{k}s}\}} = \sum_{s\vec{k}} \hbar \omega_{\vec{k}s} \, n_{\vec{k}s}$$

e sono fissate dagli interi $\{n_{\vec{k}s}\}$

$$Q = \sum_{\{n_{\vec{k}s}\}} e^{-\beta E_{\{n_{\vec{k}s}\}}} = \sum_{\{n_{\vec{k}s}\}} e^{-\sum_{s\vec{k}} \beta \hbar \omega_{\vec{k}s} n_{\vec{k}s}}$$

$$= \sum_{\{n_{\vec{k}s}\}} \prod_{s\vec{k}} e^{-\beta \hbar \omega_{s\vec{k}} n_{\vec{k}s}}$$

$$= \prod_{s\vec{k}} \sum_{n_{\vec{k}s}=0}^{\infty} e^{-\beta \hbar \omega_{\vec{k}s} n_{\vec{k}s}}$$

$$Q = \prod_{s\vec{k}} \frac{1}{1 - e^{-\beta \hbar \omega_{\vec{k}s}}}$$

② $$E = \frac{\partial \beta A}{\partial \beta} = -\frac{\partial \ln Q}{\partial \beta} = +\frac{\partial}{\partial \beta} \sum_{s\vec{k}} \ln\left[1 - e^{-\beta \hbar \omega_{\vec{k}s}}\right]$$

$$E = \sum_{s\vec{k}} \frac{e^{-\beta \hbar \omega_{\vec{k}s}}}{1 - e^{-\beta \hbar \omega_{\vec{k}s}}} \cdot \hbar \omega_{\vec{k}s}$$

$$\boxed{E = \sum_{s\vec{k}} \frac{\hbar c k}{e^{\beta \hbar c k} - 1}}$$

$$E = 2 \int_0^{k_D} \frac{d\vec{k}}{(2\pi)^2/A} \frac{\hbar c k}{e^{\beta \hbar c k} - 1}$$

$$= 2 \cdot \frac{2\pi}{(2\pi)^2/A} \int_0^{k_D} dk \, k \frac{\hbar c k}{e^{\beta \hbar c k} - 1} \qquad t = \beta \hbar c k$$

$$E = \frac{A}{\pi} \frac{(k_B T)^3}{(\hbar c)^2} \int_0^{\beta \hbar c k_D} dt \frac{t^2}{e^t - 1} \qquad \beta \hbar c k_D = \frac{T_D}{T}$$

$$\boxed{E = \frac{A}{\pi} \frac{(k_B T)^3}{(\hbar c)^2} \int_0^{T_D/T} dt \frac{t^2}{e^t - 1}}$$

$$T \to 0 \qquad \boxed{E \simeq \frac{A}{\pi} \frac{(k_B T)^3}{(\hbar c)^2} \zeta(3)}$$

3) $$c_v = \frac{\partial E/A}{\partial T} = \frac{3}{\pi} \frac{k_B^3 T^2}{(\hbar c)^2} \zeta(3)$$

4) $$N = \frac{\pi k_D^2}{(2\pi)^2/A} \qquad \boxed{k_D^2 = 4\pi \frac{N}{A} = 4\pi n}$$

$$k_D = 2\sqrt{\pi n} = 2\sqrt{3.14 \times 10^9}.$$

$$\boxed{k_D = 1.12 \times 10^5 \text{ cm}^{-1}}$$