SCRITTO DEL 24/01/12

Esercizio 1.

(1) $Q_N = \frac{1}{N!} q^N \quad , \; q = \int d\bar{r} \int d\bar{p}\, e^{-\beta h(\bar{r})} \cdot \frac{1}{h^3}$

$$q = \int_{A_b} ds \int_0^{L_z} dz\, e^{-\beta m g z} \int d\bar{p}\, e^{-\frac{\beta}{2m}\left(\bar{p} - \frac{q}{c} A(\bar{r})\right)^2} \frac{1}{h^3}$$

$-\infty \leq p_x, p_y, p_z \leq \infty$

Introduciamo le nuove variabili

$$p'_\alpha = p_\alpha - \frac{q}{c} A_\alpha(\bar{r}) ;$$

essendo $A_\alpha(\bar{r})$ limitato, vale ancora $-\infty \leq p'_\alpha \leq \infty$.
Quindi

$$q = \int_{A_b} ds \int_0^{L_z} dz\, e^{-\beta m g z} \int d\bar{p}'\, e^{-\frac{\beta p'^2}{2m}} \frac{1}{h^3}$$

$$= A_b \frac{K_B T}{mg} \left(1 - e^{-\frac{L_z m g}{K_B T}}\right) \frac{1}{\lambda^3} \qquad \lambda^2 = \frac{h^2}{2\pi m K_B T}$$

$$= A_b l \left(1 - e^{-L_z/l}\right) \frac{1}{\lambda^3} \qquad l = \frac{K_B T}{mg}$$

$$A = -K_B T \ln\left(\frac{q^N}{N!}\right) \simeq -K_B T \ln\left(\frac{e q}{N}\right)^N$$

$$\boxed{A = -N K_B T \ln\left[\frac{e}{\lambda^3 N} A_b l \left(1 - e^{-L_z/l}\right)\right]}$$

② $$\mu = \frac{\partial A}{\partial N}\Big|_{V,T} = -k_B T \ln\left[\frac{e}{\lambda^3 N} A_b \ell \left(1 - e^{-L_z/\ell}\right)\right] + k_B T$$

$$\boxed{\mu = k_B T \ln\left[\frac{\lambda^3 N}{A_b \ell} \frac{1}{1 - e^{-L_z/\ell}}\right]}$$

③ $$E = \frac{\partial \beta A}{\partial \beta}\Big|_{V,N} = -\frac{\partial}{\partial \beta} N \ln\left[\frac{1}{\beta^{3/2}} \frac{1}{\beta} \left(1 - e^{-L_z m g \beta}\right)\right]$$

$$= \frac{5}{2} N k_B T - \frac{N}{1 - e^{-L_z/\ell}} L_z m g\, e^{-L_z/\ell}$$

$$= N k_B T \left[\frac{5}{2} - \frac{(L_z/\ell)\, e^{-L_z/\ell}}{1 - e^{-L_z/\ell}}\right]$$

$$\boxed{E = N k_B T \left[\frac{5}{2} - \frac{e^{-L_z/\ell} (L_z/\ell)}{1 - e^{-L_z/\ell}}\right]}$$

④ Poiché A non dipende da B (A, l'energia libera di Helmoltz)

$$\boxed{M = -\frac{\partial A}{\partial B}\Big|_{T,V,N} = 0}$$

Esercizio 2.

① Le autofunzioni del moto libero in P.B.C. sono per la parte orbitale onde piane

$$\frac{e^{i\bar{p}\cdot\bar{r}/\hbar}}{\sqrt{A}} ,$$

con $p = \hbar k$ e

$$k_\alpha = \frac{2\pi}{L}(l, m) \qquad l, m \in \mathbb{Z} ;$$

le energie sono $\epsilon_{\bar{k}, s_z} = \hbar^2 k^2/2m$ per dato k e

$$s_z = -s, -s+1, \dots , s-1, s$$

Poiché le energie crescono con k, occupiamo i k più corti all'interno di un cerchio (nello spazio dei $\bar{k}$) che accomoda tutti e N i fermioni. k_F è il raggio del cerchio.

$$(2s+1) \frac{\pi k_F^2}{\left(\frac{2\pi}{L}\right)^2} = N$$

$$k_F^2 = 4\pi \frac{N}{A} \frac{1}{2s+1} \equiv 4\pi \frac{\rho}{2s+1}$$

$$\boxed{k_F = \sqrt{\frac{4\pi\rho}{2s+1}}}$$

(2)

$$E = \sum_{S_z} \sum_{k<k_F} \frac{\hbar^2 k^2}{2m} = (2S+1) \int \frac{d\bar{k}}{\left(\frac{2\pi}{L}\right)^2} \frac{\hbar^2}{2m} k^2$$

$$= \frac{(2S+1)A}{4\pi^2} \frac{\hbar^2}{2m} \int_{k<k_F} d\bar{k}\, k^2$$

$$\int_{k<k_F} d\bar{k}\, k^2 = 2\pi \int_0^{k_F} dk\, k\, k^2 = 2\pi \frac{k_F^4}{4} = \frac{\pi k_F^4}{2}$$

$$E = \frac{(2S+1)A}{4\pi^2} \frac{\hbar^2}{2m} \frac{\pi}{2} k_F^4 = \frac{(2S+1)A}{8\pi} \frac{\hbar^2}{2m} \left(\frac{4\pi\rho}{2S+1}\right)^2$$

$$\boxed{E = \frac{\hbar^2}{2m} \frac{2\pi\rho^2}{2S+1} \cdot A} = N \frac{\hbar^2}{2m} \frac{2\pi\rho}{2S+1}$$

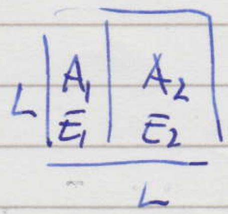
(3)

$$P = -\frac{\partial E}{\partial A} = -\frac{\partial}{\partial A}\left[N \frac{\hbar^2}{2m} \frac{2\pi\rho}{2S+1} \right] =$$

$$= -N \frac{\hbar^2}{2m} \frac{2\pi}{2S+1} \frac{\partial}{\partial A}\rho = \frac{\hbar^2}{2m} \frac{2\pi\rho^2}{2S+1}$$

$$\boxed{P = \frac{\hbar^2}{2m} \frac{2\pi\rho^2}{2S+1}} \qquad \rho = \frac{N}{A}$$

(4)



$$P_1 = -\frac{\partial E_1}{\partial A_1} = \frac{\hbar^2}{2m} \frac{2\pi\rho_1^2}{2S_1+1}$$

$$P_2 = -\frac{\partial E_2}{\partial A_2} = \frac{\hbar^2}{2m} \frac{2\pi\rho_2^2}{2S_2+1}$$

$$A_1 + A_2 = A$$

$$P_1 = P_2 \Rightarrow \boxed{\frac{\rho_2}{\rho_1} = \sqrt{\frac{2S_2+1}{2S_1+1}} = \sqrt{2}} \quad \boxed{\text{per } S_2 = \frac{3}{2},\ S_1 = \frac{1}{2}}$$