Exercise 1 9-05-14 $\hbar = m = e = 1$

(1)
$$\epsilon_\sigma = \frac{\sum_{|k_\sigma| < k_{F\sigma}} v_F |k|}{\sum_{|k_\sigma| < k_{F\sigma}}} = \frac{v_F \int_0^{k_{F\sigma}} dk\, k\, k}{\int_0^{k_{F\sigma}} dk\, k} = \frac{v_F k_{F\sigma}^3/3}{k_{F\sigma}^2/2} = \frac{2}{3} v_F k_{F\sigma}$$

$$\epsilon_\sigma = \frac{2}{3}\epsilon_{F\sigma} \qquad \epsilon_{F\sigma} = v_F k_{F\sigma}$$

$$\frac{\pi k_{F\sigma}^2}{\frac{(2\pi)^2}{A}} = N_\sigma = \frac{k_{F\sigma}^2}{4\pi} A \Rightarrow \boxed{k_{F\sigma}^2 = 4\pi \frac{N_\sigma}{A} = 4\pi n_\sigma}$$

$$t = \frac{\sum_\sigma N_\sigma \epsilon_\sigma}{N} = \frac{\sum_\sigma n_\sigma \epsilon_\sigma}{n} = \sum_\sigma \frac{2}{3} v_F \frac{2\sqrt{\pi n_\sigma}}{n} n_\sigma$$

$$t = c\left[\frac{n_\uparrow^{3/2}}{n} + \frac{n_\downarrow^{3/2}}{n}\right] \qquad \boxed{c = \frac{4}{3}\sqrt{\pi}\, v_F}$$

(2)
$$e(n,\zeta,B) = \frac{1}{N}\left[N t + \frac{U N_\uparrow N_\downarrow}{N} + \mu_B B (N_\uparrow - N_\downarrow)\right]$$

$$= t + U\frac{n_\uparrow n_\downarrow}{n^2} + \frac{\mu_B B}{n}(n_\uparrow - n_\downarrow)$$

$$e(n,\zeta,B) = \frac{c\sqrt{n}}{2^{3/2}}\left[(1+\zeta)^{3/2} + (1-\zeta)^{3/2}\right] + \frac{1}{4}U(1-\zeta^2) + \mu_B B \zeta$$

③ $\zeta \ll 1$

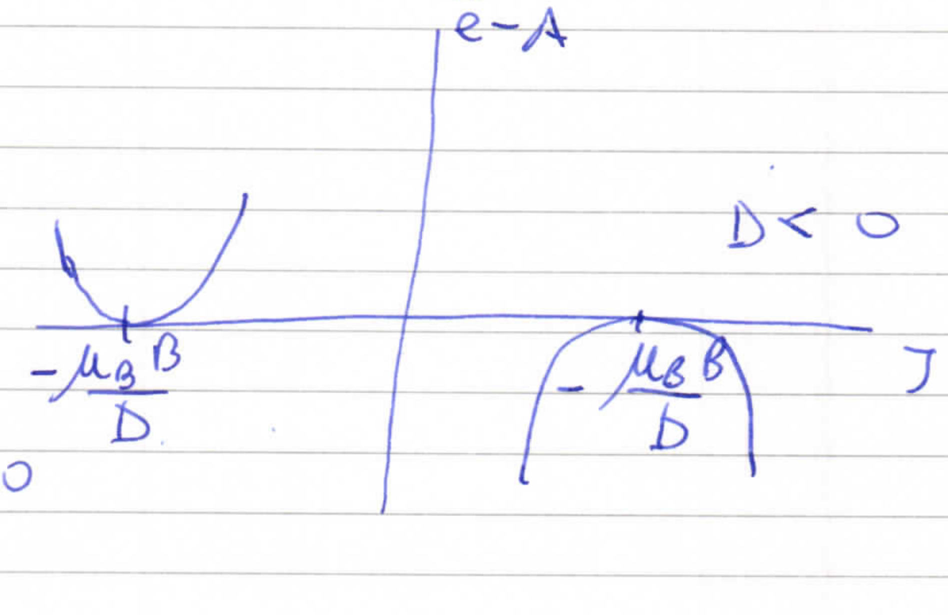
$$e(n,\zeta,B) \simeq \frac{c}{\sqrt{2}}\sqrt{n}\left(1+\frac{3}{8}\zeta^2\right) + \frac{1}{4}U\left(1-\zeta^2\right) + \mu_B B \zeta$$
$$\equiv \tilde{e}(n,\zeta,B)$$

④ $$\tilde{e}(n,\zeta,B) = \frac{c}{\sqrt{2}}\sqrt{n} + \frac{U}{4} + \mu_B B\zeta + \left(\frac{3\sqrt{2n}}{16}c - \frac{U}{4}\right)\zeta^2$$

$$= \frac{c}{\sqrt{2}}\sqrt{n} + \frac{U}{4} + \left(\frac{3c}{16}\sqrt{2n} - \frac{U}{4}\right)\left(\zeta^2 + 2\zeta\frac{\mu_B B}{\frac{3c\sqrt{2n}}{8} - \frac{U}{2}}\right)$$

$$= \frac{c}{\sqrt{2}}\sqrt{n} + \frac{U}{4} + \left(\frac{3c}{16}\sqrt{2n} - \frac{U}{4}\right)\left(\zeta + \frac{\mu_B B}{\frac{3c\sqrt{2n}}{8} - \frac{U}{2}}\right)^2$$
$$- \frac{\mu_B^2 B^2}{\frac{3c\sqrt{2n}}{4} - \frac{U}{2}}$$

$$\tilde{e}(n,\zeta,B) = A(n,B) + \frac{1}{2}D(n)\left(\zeta + \frac{\mu_B B}{D}\right)^2$$



$D > 0 \Rightarrow 3C\sqrt{2n} > 4U$

$$\tilde{e}_{min} = \frac{C}{\sqrt{2}}\sqrt{n} + U - \frac{\mu_B^2 B^2}{\frac{3C}{4}\sqrt{2n} - \frac{U}{2}}$$

$$\zeta_{min} = -\frac{\mu_B B}{D}, \quad B < \frac{D}{\mu_B}$$

$$= -1, \quad B > \frac{D}{\mu_B}$$

$D < 0 \quad 3C\sqrt{2n} < 4U$

$$\tilde{e}_{min} = \frac{11}{8}\frac{C\sqrt{n}}{\sqrt{2}} - \mu_B B$$

$$\zeta_{min} = -1$$

(5)

$$3C\sqrt{2n} \gg 4U, \quad D > 0, \quad D \simeq \frac{1}{\frac{3C\sqrt{2n}}{8}}$$

$$\zeta \simeq -\frac{\mu_B}{\frac{3C\sqrt{2n}}{8}} B \qquad B < \frac{3C\sqrt{2n}}{8\mu_B}$$

(6) $3C\sqrt{2n} \ll 4U, \quad D < 0, \quad D \simeq -\frac{2}{U}$

$$\zeta \simeq -1$$

Exercise 2   9-05-14

(1) An insulator $\boxed{\Delta > 0}$

(2) $E_v(q) = -2t_v(-1 - \frac{q^2 a^2}{2})$

$E_c(q) = -2t_c(-1 - q^2\frac{a^2}{2})$

(3) $\rho_v(E) = \frac{2}{A}\sum_{q} \delta(E + 2t_v + t_v q^2 a^2)$

$= \frac{2}{A}\int \frac{d^2 q}{\frac{(2\pi)^2}{A}} \delta(E + 2t_v + t_v q^2 a^2)$

$= \frac{1}{\pi}\int_0 dq\, q\, \delta(E + 2t_v + t_v q^2 a^2)$

$= \frac{1}{2\pi}\int_0 dy\, \delta(E + 2t_v + t_v a^2 y)$

$= \frac{1}{2\pi}\frac{1}{t_v a^2}\theta(-E - 2t_v)$

$\rho_c(E) = \frac{1}{2\pi t_c a^2}\theta(E - 2t_c)$

(4)

$$n_c = \int_0^\infty g_c(\epsilon)\, e^{-\beta(\epsilon-\epsilon_c)}\, d\epsilon =$$

$$= \int_{2\gamma_c}^{\infty} d\epsilon\, \frac{1}{2\pi\gamma_c a^2}\, e^{-\beta(\epsilon-2\gamma_c)}$$

$$= \frac{k_B T}{2\pi\gamma_c a^2}$$

$$p_v = \int_{-\infty}^{0} g_v(\epsilon)\, e^{-\beta(\epsilon-\epsilon_v)}\, d\epsilon$$

$$= \int_{-\infty}^{-2\gamma_v} d\epsilon\, \frac{1}{2\pi\gamma_v a^2}\, e^{+\beta(\epsilon+2\gamma_v)}$$

$$= \frac{k_B T}{2\pi\gamma_v a^2}$$

(5)

$$1 = \frac{n_c}{p_v} = \frac{N_c}{P_v}\, e^{-\beta(\epsilon_c+\epsilon_v-2\mu)} = e^{-\beta(2\gamma_c-2\gamma_v-2\mu)}\, \frac{\gamma_v}{\gamma_c}$$

$$\Rightarrow \boxed{\mu = \gamma_c - \gamma_v + \frac{k_B T}{2} \ln \frac{\gamma_c}{\gamma_v}}$$

(6)

$$\frac{\gamma_c}{\gamma_v} = 1$$