Written exam, 24-02-2012

Exercise 1.

$$U = \frac{K}{4d^2} \sum_{\bar{R}}' \sum_{\bar{d}}' [(\bar{u}_{\bar{R}+\bar{d}} - \bar{u}_{\bar{R}})\cdot \bar{d}]^2$$

① $$\frac{\partial U}{\partial u_{\alpha \bar{R}'}} = \frac{K}{2d^2} \sum_{\bar{d}}' d_\alpha (\bar{u}_{\bar{R}} - \bar{u}_{\bar{R}+\bar{d}} + \bar{u}_{\bar{R}} - \bar{u}_{\bar{R}-\bar{d}})\cdot \bar{d}$$

$$= \frac{K}{2d^2} \sum_{\bar{d}}' d_\alpha (2\bar{u}_{\bar{R}'} - \bar{u}_{\bar{R}'+\bar{d}} - u_{\bar{R}'-\bar{d}})\cdot \bar{d}$$

② $$\frac{\partial^2 U}{\partial u_{\beta \bar{R}''}\partial u_{\alpha \bar{R}'}} = \frac{K}{2d^2} \sum_{\bar{d}}' d_\alpha d_\beta (2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'+\bar{d},\bar{R}''} - \delta_{\bar{R}'-\bar{d},\bar{R}''})$$

③ $$\frac{\bar{d}}{d} = (1,0),\ (-1,0),\ (0,1),\ (0,-1)$$

$$D_{xx} = \frac{K}{2}[2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'+\hat{x}a,\bar{R}''} - \delta_{\bar{R}'-\hat{x}a,\bar{R}''}$$
$$+ 2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'-\hat{x}a,\bar{R}''} - \delta_{\bar{R}'+\hat{x}a,\bar{R}''}]$$

$$D_{xx}(\bar{R}'-\bar{R}'') = K[2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'+\hat{x}a,\bar{R}''} - \delta_{\bar{R}'-\hat{x}a,\bar{R}''}]$$

$$D_{yy}(\bar{R}'-\bar{R}'') = K[2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'+\hat{y}a,\bar{R}''} - \delta_{\bar{R}'-\hat{y}a,\bar{R}''}]$$

$$D_{xy}(\bar{R}'-\bar{R}'') = D_{yx}(\bar{R}'-\bar{R}'') = 0$$

Note if $d_x \neq 0 \Rightarrow d_y = 0$; if $d_y \neq 0 \Rightarrow d_x = 0$.

(4)

$$D_{xx}(\bar{q}) = \sum_{\bar{R}''}' D_{xx}(\bar{R}'-\bar{R}'') e^{-i\bar{q}(\bar{R}'-\bar{R}'')}$$

$$= K \sum_{\bar{R}''}' [2\delta_{\bar{R}',\bar{R}''} - \delta_{\bar{R}'+\hat{x}a,R''} - \delta_{\bar{R}'-\hat{x}a,\bar{R}''}] e^{-i\bar{q}(R'-R'')}$$

$$D_{xx}(q) = K[2 - e^{iq_x a} - e^{-iq_x a}] = 4\sin^2\left(\frac{q_x a}{2}\right)$$

$$D_{yy}(q) = K[2 - e^{iq_y a} - e^{-iq_x a}] = 4\sin^2\left(\frac{q_y a}{2}\right)$$

$$D_{xy}(q) = D_{yx}(q) = 0$$

(5)

$$\sum_{\beta} D_{\alpha\beta}(\bar{q}) \epsilon_\beta = \lambda_\gamma \epsilon_\alpha^{(r)} \Rightarrow \begin{pmatrix} D_{xx}(\bar{q}) & 0 \\ 0 & D_{yy}(\bar{q}) \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$$

$$\lambda_1 = D_{xx}(\bar{q}) = 4K\sin^2\left(\frac{q_x a}{2}\right)$$

$$\bar{\epsilon}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = D_{yy}(\bar{q}) = 4K\sin^2\left(\frac{q_y a}{2}\right) \qquad \bar{\epsilon}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(6)

$$\sum_{\beta}' D_{\alpha\beta}(\bar{q})\, \epsilon_{\beta}(\bar{q}) = M\omega^2(q)\, \epsilon_{\alpha}(\bar{q})$$

$$\Rightarrow \quad \omega(\bar{q}) = \sqrt{\frac{\lambda(\bar{q})}{M}} =$$

$$\omega_1(\bar{q}) = 2\sqrt{\frac{K}{M}}\left|\sin\left(\frac{q_x a}{2}\right)\right| = \omega_0 \left|\sin\left(\frac{q_x a}{2}\right)\right|$$

$$\omega_2(\bar{q}) = 2\sqrt{\frac{K}{M}}\left|\sin\left(\frac{q_y a}{2}\right)\right| = \omega_0 \left|\sin\left(\frac{q_y a}{2}\right)\right|$$

$$g(\omega) = \frac{1}{A}\sum_{\alpha=1}^{2}{}' \sum_{\bar{k}\in F.B.Z} \delta(\omega - \omega_\alpha(\bar{k})) = \sum_{\alpha}' g_\alpha(\omega)$$

$$g_1(\omega) = \frac{1}{A}\frac{A}{(2\pi)^2}\int_{-\frac{\pi}{a}}^{\pi/a} dq_x\, \delta\left(\omega - \omega_0\left|\sin\left(\frac{q_x a}{2}\right)\right|\right)\int_{-\frac{\pi}{a}}^{\pi/a} dq_y$$

$$= \frac{1}{(2\pi)^2}\frac{2\pi}{a}\cdot 2\int_0^{\pi/a} dq_x\, \delta\left(\omega - \omega_0 \sin\left(\frac{q_x a}{2}\right)\right)$$

$$= \frac{1}{\pi a}\frac{2}{a}\int_0^{\pi/2} dy\, \delta(\omega - \omega_0 \sin y)$$

$$\omega - \omega_0 \sin y_0 = 0 \quad \Rightarrow \quad \sin y_0 = \frac{\omega}{\omega_0} \Rightarrow \cos y_0 = \sqrt{1 - \frac{\omega^2}{\omega_0^2}}$$

$$g_1(\omega) = \frac{2}{\pi a^2} \frac{1}{|\omega_0 \cos\varphi_0|} = \frac{2}{\pi a^2} \frac{1}{\sqrt{\omega_0^2 - \omega^2}} = g_2(\omega)$$

$$g(\omega) = \frac{4}{\pi a^2} \frac{1}{\sqrt{\omega_0^2 - \omega^2}} \qquad 0 \leq \omega \leq \omega_0$$

Esercizio 2.

①

$$g_\sigma(E,H) = \frac{1}{V}\sum_{\vec{k}} \delta\left(E - \frac{\hbar^2}{2m}k^2 - \sigma\mu_B H\right)$$

$$= \frac{1}{V}\sum_{\vec{k}} \delta\left([E-\sigma\mu_B H] - \frac{\hbar^2}{2m}k^2\right)$$

$$g_\sigma(E,H) = \frac{1}{2} g_0(E - \sigma\mu_B H)$$

②

$$n_\sigma(T,H,\mu) = \int dE \frac{g_\sigma(E,H)}{e^{\beta(E-\mu)}+1} = \frac{1}{2}\int_{\sigma\mu_B H}^{\infty} dE \frac{g_0(E-\sigma\mu_B H)}{e^{\beta(E-\mu)}+1}$$

set $E' = E - \sigma\mu_B H$, $dE' = dE$

$$n_\sigma(T,H,\mu) = \frac{1}{2}\int_0^{\infty} dE' \frac{g_0(E')}{e^{\beta(E'-\mu+\sigma\mu_B H)}}$$

$$= n_\sigma(T,0,\mu-\sigma\mu_B H) \equiv n_\sigma(T,0,\tilde{\mu}_\sigma)$$

③

$$n_\sigma(T,H,\mu) = n_\sigma(T,0,\tilde{\mu}_\sigma) = \frac{1}{2}\int_0^{\infty} dE \frac{g_0(E)}{e^{\beta(E-\tilde{\mu}_\sigma)}+1}$$

$$= \frac{1}{2}\left[\int_0^{\tilde{\mu}_\sigma} dE\, g_0(E) + \frac{\pi^2}{6}(k_B T)^2 g_0'(\tilde{\mu}_\sigma) + \dots\right]$$

④

$$M = -\frac{\mu_B}{2}\left[\int_0^{\mu-\mu_B H} dE\, g_0(E) - \int_0^{\mu+\mu_B H} dE\, g_0(E) + \frac{\pi^2}{6}(k_B T)^2 \times \right.$$

$$\left. \times \left[g_0'(\mu-\mu_B H) - g_0'(\mu+\mu_B H)\right]\right]$$

$$M = +\frac{\mu_B}{2}\left\{\int_{\mu-\mu_B H}^{\mu+\mu_B H} dE\, g_0(E) + \frac{\pi^2}{6}(k_B T)^2\left[g_0'(\mu+\mu_B H) - g_0'(\mu-\mu_B H)\right]\right\}$$

(5) • To terms linear in H, $\mu(T,H) = \mu_0(T)$

• $\int_{\mu-\mu_B H}^{\mu+\mu_B H} dE\, g_0(E) = \int_{\mu_0-\mu_B H}^{\mu_0+\mu_B H} dE\, g_0(E) \simeq [\mu_0+\mu_B H - (\mu_0 - \mu_B H)]\, g_0(\mu_0)$

$= 2\mu_B H\, g_0(\mu_0)$

• $g_0'(\mu+\mu_B H) - g_0'(\mu - \mu_B H) \simeq g_0''(\mu)\, 2\mu_B H \simeq -\frac{1}{2}\frac{g_0(\mu_0)}{\mu_0^2}\mu_B H$

$g_0(E) \sim E^{\frac{1}{2}} \quad g_0''(E) = -\frac{1}{4}\frac{g_0(E)}{E^2}$

$M = \frac{\mu_B}{2}\left[2\mu_B H\, g_0(\mu_0) + \frac{\pi^2}{6}(k_B T)^2\left(-\frac{1}{2}\frac{g_0(\mu_0)}{\mu_0^2}\mu_B H\right)\right]$

$= \mu_B^2 g_0(\mu_0)\cdot H\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\mu_0}\right)^2\right]$

$$M = \mu_B^2 g_0(\mu_0)\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\mu_0}\right)^2\right]\cdot H$$

(6) $\chi_s = \mu_B^2 g_0(\mu_0)\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\mu_0}\right)^2\right]$

• recalling $\mu_0(T) = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$ and

$g_0(E) = C\sqrt{E}$, we get

$g_0(\mu_0) = C\sqrt{\epsilon_F}\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]^{\frac{1}{2}} \simeq C\sqrt{\epsilon_F}\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$

$= g_0(\epsilon_F)\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$

• also $\left(\frac{k_B T}{\mu_0}\right)^2 \simeq \left(\frac{k_B T}{\epsilon_F}\right)^2 + O(T^4)$

• finally

$$\chi_s = \mu_B^2 g_0(\epsilon_F)\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]\left[1 - \frac{\pi^2}{24}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$$

$$\chi_s \simeq \mu_B^2 g_0(\epsilon_F)\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$$