Exercise I   11.06.15

① $\hat{H} = -\mu_B g H \hat{S}_z \Rightarrow E_\sigma = \frac{\mu_B g H}{2}\sigma$, $\sigma = \pm 1$

$= \mu_B H \sigma$ $g = 2$

$$Z = e^{\beta\mu_B H} + e^{-\beta\mu_B H} = 2\cosh(\beta\mu_B H)$$

$$F = -k_B T \ln[2\cosh(\beta\mu_B H)]$$

② $M = -\frac{\partial F}{\partial H} = k_B T \beta\mu_B \,\mathrm{tgh}(\beta\mu_B H)$

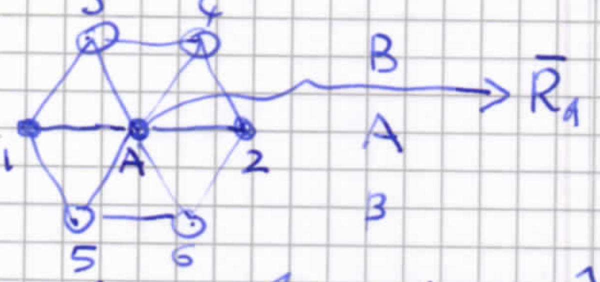
$$M = \mu_B \,\mathrm{tgh}(\beta\mu_B H)$$

③ $\hat{\mathcal{H}} = \frac{1}{2} k_B T_N \sum_{\bar{R},\bar{d}} \hat{\vec{S}}(\bar{R}) \cdot \hat{\vec{S}}(\bar{R}+\bar{d}) - \mu_B g H \sum_{\bar{R}} \hat{S}_z(R)$

$\bar{d}$ = nearest neighbors :
$(\pm a, 0)$, $a\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

④

1) site A



$k_B T_N \hat{\vec{S}}(\bar{R}_A) \cdot [\hat{\vec{S}}(\bar{R}_1) + \hat{\vec{S}}(\bar{R}_2) + \hat{\vec{S}}(\bar{R}_3) + \hat{\vec{S}}(\bar{R}_4) + \hat{\vec{S}}(\bar{R}_5) + \hat{\vec{S}}(\bar{R}_6)]$

$\simeq k_B T_N \hat{\vec{S}}(\bar{R}_A)[\langle \quad \rangle]$ $\boxed{g=2}$

$= k_B T_N \hat{\vec{S}}(\bar{R}_A) \cdot \hat{q} \left[\frac{2}{n\mu_B g}\right][2M_A + 4M_B]$

$-\mu_B H g \hat{S}_4 + \frac{k_B T_N}{n\mu_B}[2M_A + 2M_B]\hat{S}_4 = -\mu_B h_A g \hat{S}_4$

$$h_A = H - \frac{2k_B T_N}{n\mu_B^2 g}[M_A + 2M_B]$$

2) Interchange A and B to obtain

$$h_B = H - \frac{k_B T_N}{n\mu_B^2}[M_B + 2M_A]$$

(5)

$$M_A = \frac{n}{2}\mu_B \, tgh(\beta h_A \mu_B)$$

$$H=0 \rightarrow \quad M_A = \frac{n}{2}\mu_B \, tgh\left(-\beta \frac{k_B T_N}{n\mu_B^{\cancel{2}}}[M_A + 2M_B]\cancel{\mu_B}\right)$$

$$M_A = -\frac{n}{2}\mu_B \, tgh\left(\frac{1}{n\mu_B}\frac{T_N}{T}[M_A + 2M_B]\right)$$

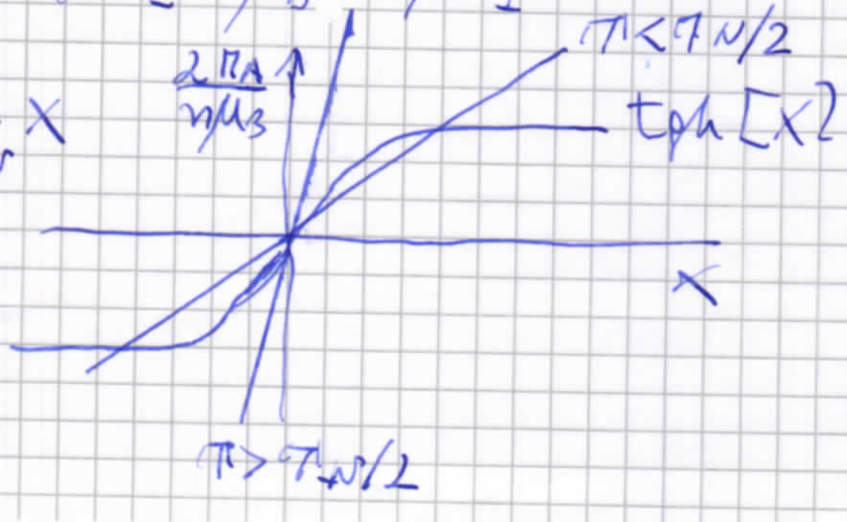
$$M_B = -\frac{n}{2}\mu_B \, tgh\left(\frac{1}{n\mu_B}\frac{T_N}{T}[M_B + 2M_A]\right)$$

(6) $M_B = -M_A$

$$M_A = \frac{n}{2}\mu_B \, tgh\left(\frac{1}{n\mu_B} M_A \frac{T_N}{T}\right)$$

$$\frac{2M_A}{n\mu_B} = tgh\left[\frac{M_A}{n\mu_B}\frac{T_N}{T}\right] = tgh[x]$$

$$2\frac{M_A}{n\mu_B} = 2\frac{T}{T_N}x$$



$$\boxed{T_c = T_N/2}$$

Exercise II 11.06.15

(1)

$$\frac{E}{V} = \frac{1}{V}\sum_{s,\vec{k}\in F.BZ} \frac{\hbar\omega_s(\vec{k})}{e^{\beta\hbar\omega_s(\vec{k})}-1}$$

$$\frac{E}{V} = \int d\omega \, \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}\, g(\omega) \qquad (1)$$

$$g(\omega) = \frac{1}{V}\sum_{s,\vec{k}\in F.BZ.} \delta(\omega - \omega_s(\vec{k}))$$

(2) The integrand in eq. (1) for $\hbar\omega \gg k_B T$ behaves as $e^{-\beta\hbar\omega}$, giving exponentially small contributes.

Sizeable contributes come from $\hbar\omega \lesssim k_B T$.

For $k_B T \to 0$ only small values of $\hbar\omega$ are relevant.

(3)

$$g(\omega) \simeq \frac{1}{V}\sum_{s=1}^{D}\sum_{\vec{k}} \delta\left(\omega - \omega_{0s}\left(\frac{k}{q_s}\right)^{\nu}\right), \quad k \to 0$$

$$\simeq \sum_{s=1}^{D}\int \frac{d\vec{k}}{(2\pi)^D}\, \delta\left(\omega - \omega_{0s}\left(\frac{k}{q_s}\right)^{\nu}\right)$$

$$= \sum_{s=1}^{D}\frac{\Omega_D}{(2\pi)^D}\int_0^{\infty} dk\, k^{D-1}\, \delta\left(\omega - \omega_{0s}\left(\frac{k}{q_s}\right)^{\nu}\right)$$

$$y = \frac{k}{q_s}$$

$$g(\omega) = \frac{\Omega_D}{(2\pi)^D} \sum_{s=1}^{D} q_s^D \int_0^\infty dy\, y^{D-1}\, \delta(\omega - \omega_{0s} y^\nu)$$

$$y_\omega = \left(\frac{\omega}{\omega_{0s}}\right)^{\frac{1}{\nu}}$$

$$= \frac{\Omega_D}{(2\pi)^D} \sum_{s=1}^{D} \frac{q_s^D\, y_\omega^{D-1}}{\omega_{0s}\, \nu\, y_\omega^{\nu-1}}$$

$$= \frac{\Omega_D}{(2\pi)^D} \sum_{s=1}^{D} \frac{q_s^D}{\omega_{0s}\nu}\, y_\omega^{D-\nu} = \frac{\Omega_D}{(2\pi)^D} \sum_{s=1}^{D} \frac{q_s^D}{\nu} \frac{\omega^{\frac{D}{\nu}-1}}{\omega_{0s}^{D/\nu}}$$

$$g(\omega) = \left[\frac{\Omega_D}{(2\pi)^D \nu} \sum_{s=1}^{D} \frac{q_s^D}{\omega_{0s}^{D/\nu}}\right] \omega^{\frac{D}{\nu}-1} \equiv C_1\, \omega^{\frac{D}{\nu}-1}$$

(4) $$\frac{E}{V} = C_1 \int_0^\infty d\omega\, \frac{\hbar\omega\, \omega^{\frac{D}{\nu}-1}}{e^{\beta\hbar\omega}-1}$$

As $\hbar\omega \gg k_B T$ gives exponentially small contributions, the integral can be extended to ∞!

(5) $$\frac{E}{V} = C_1 \frac{(k_B T)^{\frac{D}{\nu}+1}}{\hbar^{\frac{D}{\nu}}} \int_0^\infty dt\, \frac{t^{D/\nu}}{e^t - 1} \equiv B$$

$$\frac{E}{V} = \frac{B \cdot C_1}{\hbar^{D/\nu}} (k_B T)^{\frac{D}{\nu}+1}$$

The integrand behaves as $t^{\frac{D}{\nu}-1}$ for $t \to 0$ and $t^{\frac{D}{\nu}} e^{-t}$ for $t \to \infty$: no convergence problem!

6 For $T \to 0$

$$C_V = \left(\frac{D}{\nu}+1\right) \frac{B\, C_1}{\hbar^{D/\nu}} k_B^{\frac{D}{\nu}+1}\, T^{D/\nu} \sim T^{\frac{D}{\nu}}$$