Esercizio 1. 5/11/09

1.
$$Q_N = \frac{1}{h^{3N} N!}\left[\int d\bar{p}\, e^{-\beta p^2/2m} \int_{A_b} ds \int_0^{L_z} dz\, e^{-\beta m g z}\right]^N$$

$$= \frac{(2\pi m K_B T)^{3N/2}}{h^{3N}\, N!}\left[A_b \frac{K_B T}{mg}\left(1-e^{-\beta m g L_z}\right)\right]^N$$

Ponendo $\lambda^2 = h^2/(2\pi m K_B T)$ e $\ell = K_B T/(mg)$

$$Q_N = \frac{1}{\lambda^{3N} N!}\left[A_b \ell \left(1-e^{-L_z/\ell}\right)\right]^N$$

$$\simeq \left(\frac{e}{\lambda^3 N} A_b \ell \left(1-e^{-L_z/\ell}\right)\right)^N$$

$$A = -N K_B T \ln\left[\frac{e}{\lambda^3 N} A_b \ell \left(1-e^{-L_z/\ell}\right)\right]$$

2.
$$P = -\frac{\partial A}{\partial V} = -\frac{1}{A_b}\frac{\partial A}{\partial L_z} = \frac{1}{A_b} N K_B T \frac{\frac{1}{\ell} e^{-L_z/\ell}}{1-e^{-L_z/\ell}}$$

$$= \frac{N K_B T}{A_b L_z}\frac{L_z}{\ell}\frac{e^{-L_z/\ell}}{1-e^{-L_z/\ell}} = K_B T \frac{N}{V}\cdot\frac{L_z}{\ell}\frac{e^{-L_z/\ell}}{1-e^{-L_z/\ell}}$$

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3. $$\mu = \frac{\partial A}{\partial N} = -K_B T \ln\left[\frac{e}{\lambda^3 N} A_b \ell \left(1-e^{-L_z/\ell}\right)\right] + K_B T$$

$$= K_B T \ln\left[\frac{\lambda^3 N}{V}\frac{L_z}{\ell}\frac{1}{1-e^{-L_z/\ell}}\right]$$

4.

$$\rho(z) = N\langle \delta(r-r_1)\rangle = N \frac{\int d\bar{r}_1 e^{-\beta m g z_1} \delta(r-r_1)}{\int d\bar{r}_1 e^{-\beta m g z_1}}$$

$$= N \frac{e^{-\beta m g z}}{\int_{A_b} ds \int_0^{L_z} dz_1 e^{-\beta m g z_1}} = \frac{N e^{-\beta m g z}}{A_b \frac{k_B T}{mg}\left(1-e^{-\beta m g L_z}\right)}$$

$$\rho(z) = \frac{N}{V} \frac{L_z}{\ell} \frac{e^{-z/\ell}}{1-e^{-L_z/\ell}}$$

$$P = k_B T \frac{N}{V} \frac{L_z}{\ell} \frac{e^{-L_z/\ell}}{1-e^{-L_z/\ell}} = k_B T \rho(L_z)$$

$$\mu = k_B T \ln\left[\frac{\lambda^3 N}{V} \frac{L_z}{\ell} \frac{1}{1-e^{-L_z/\ell}}\right]$$

$$= k_B T \ln\left[\lambda^3 \rho(0)\right]$$

Esercizio 2. 5/11/09

1.

$$Q_N = \frac{1}{h^{3N}N!}\int dp\,dq\; e^{-\beta\mathcal{H}} \equiv \sum e^{-\beta\mathcal{H}}$$

$$Q_N = \sum e^{-\beta\mathcal{H}_0} e^{-\beta\lambda\mathcal{H}_1} = \sum e^{-\beta\mathcal{H}_0}\,\frac{\sum e^{-\beta\mathcal{H}_0}e^{-\beta\lambda\mathcal{H}_1}}{\sum e^{-\beta\mathcal{H}_0}}$$

$$= Q_{N,0}\langle e^{-\beta\lambda\mathcal{H}_1}\rangle_0$$

$$Q_N \simeq Q_{N,0}\langle 1-\beta\lambda\mathcal{H}_1\rangle_0 = Q_{N,0} - Q_{N,0}\langle\beta\lambda\mathcal{H}_1\rangle_0$$

2.

$$A = -k_BT\ln\left[Q_{N,0}(1-\beta\lambda\langle\mathcal{H}_1\rangle_0)\right]$$

$$= -k_BT\ln Q_{N,0} - k_BT\ln(1-\beta\lambda\langle\mathcal{H}_1\rangle_0)$$

$$\simeq A_0 - k_BT\left[-\beta\lambda\langle\mathcal{H}_1\rangle_0\right] = A_0 + \lambda\langle\mathcal{H}_1\rangle_0$$

3.

$$U = -\frac{\partial\ln Q_N}{\partial\beta} \simeq -\frac{\partial\ln\left[Q_{N,0}(1-\beta\lambda\langle\mathcal{H}_1\rangle_0)\right]}{\partial\beta}$$

$$= -\frac{\partial\ln Q_{N,0}}{\partial\beta} - \frac{\partial}{\partial\beta}\ln\left[1-\beta\lambda\langle\mathcal{H}_1\rangle_0\right]$$

$$\simeq U_0 - \frac{\partial}{\partial\beta}\left[-\beta\lambda\langle\mathcal{H}_1\rangle_0\right]$$

$$= U_0 + \lambda\langle\mathcal{H}_1\rangle_0 + \beta\lambda\frac{\partial\langle\mathcal{H}_1\rangle_0}{\partial\beta}$$

Nota: per $|x| \ll 1$

$e^x \simeq 1+x$; $\ln(1+x) \simeq x$.

$$\frac{\partial \langle \mathcal{H}_1 \rangle_0}{\partial \beta} = \frac{\partial}{\partial \beta} \frac{\sum e^{-\beta \mathcal{H}_0} \mathcal{H}_1}{\sum_i e^{-\beta \mathcal{H}_0}} = \frac{\sum e^{-\beta \mathcal{H}_0} (-\mathcal{H}_0 \mathcal{H}_1)}{\sum e^{-\beta \mathcal{H}_0}}$$

$$+ \frac{\sum_i e^{-\beta \mathcal{H}_0} \mathcal{H}_1 \sum_i e^{-\beta \mathcal{H}_0} \mathcal{H}_0}{(\sum_i e^{-\beta \mathcal{H}_0})^2}$$

$$= - \langle \mathcal{H}_0 \mathcal{H}_1 \rangle_0 + \langle \mathcal{H}_0 \rangle \langle \mathcal{H}_1 \rangle_0$$

$$U = U_0 + \lambda \langle \mathcal{H}_1 \rangle_0 + \lambda \beta \left[\langle \mathcal{H}_0 \rangle \langle \mathcal{H}_1 \rangle_0 - \langle \mathcal{H}_0 \mathcal{H}_1 \rangle_0 \right]$$

$$= U_0 + \lambda \langle \mathcal{H}_1 \rangle_0 - \lambda \beta \left[\langle (\mathcal{H}_0 - \langle \mathcal{H}_0 \rangle_0)(\mathcal{H}_1 - \langle \mathcal{H}_1 \rangle_0) \rangle_0 \right]$$

4. Utilizzando la 2. e $S = -\frac{\partial A}{\partial T} = \frac{\beta}{T} \frac{\partial A}{\partial \beta}$

$$S = -\frac{\partial A}{\partial T} = -\frac{\partial A_0}{\partial T} - \lambda \frac{\partial \langle \mathcal{H}_1 \rangle_0}{\partial T}$$

$$= S_0 + \lambda \frac{\beta}{T} \frac{\partial \langle \mathcal{H}_1 \rangle_0}{\partial \beta}$$

$$= S_0 + \lambda k_B \beta^2 \left[- \langle (\mathcal{H}_0 - \langle \mathcal{H}_0 \rangle_0)(\mathcal{H}_1 - \langle \mathcal{H}_1 \rangle_0) \rangle_0 \right]$$

$$S = S_0 - \lambda k_B \beta^2 \langle (\mathcal{H}_0 - \langle \mathcal{H}_0 \rangle_0)(\mathcal{H}_1 - \langle \mathcal{H}_1 \rangle_0) \rangle_0$$