II COMPITO 20-12-13

ESERCIZIO 1

① $$E(z,\beta,V) = \sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{H}\right) \Big/ \sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)$$

$$-\frac{\partial E}{\partial \beta}\Big|_{z,V} = \sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{H}^2\right) \Big/ \sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)$$

$$-\left[\sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\hat{H}\right)\right]^2 \Big/ \left[\sum_{N=0}^{\infty} z^N \mathrm{Tr}\left(e^{-\beta\hat{H}}\right)\right]^2$$

$$= \langle \hat{H}^2 \rangle - \langle H \rangle^2$$

② $\boxed{V = L \times L\,!}$

$$\frac{1}{V} E(z,\beta,V) = \frac{1}{V} \sum_{\bar{p}} \frac{p^2}{2m} \frac{1}{\frac{e^{\beta p^2/2m}}{z}+1}$$

$$= \frac{1}{h^2} \int d\bar{p}\, \frac{(p^2/2m)\, z\, e^{-\beta\frac{p^2}{2m}}}{1+z e^{-\beta\frac{p^2}{2m}}}$$

$$= \frac{2\pi}{h^2} \int_0^{\infty} dp\, p\, \frac{p^2}{2m} \sum_{n=1}^{\infty} (-1)^{n-1} \left(z e^{-\beta\frac{p^2}{2m}}\right)^n$$

$$= \frac{2\pi m}{h^2} \frac{1}{\beta^2} \int_0^{\infty} dx\, x \sum_{n=1}^{\infty} (-1)^{n-1} \left(z e^{-x}\right)^n \qquad x = \beta\frac{p^2}{2m}$$

$$= \frac{2\pi m}{h^2} \frac{1}{\beta^2} \sum_{n=1}^{\infty} (-1)^{n-1} z^n \int_0^{\infty} dx\, x e^{-nx}$$

$$\int_0^\infty dx\, x\, e^{-\alpha x} = -\frac{\partial}{\partial\alpha}\int_0^\infty dx\, e^{-\alpha x} = -\frac{\partial}{\partial\alpha}\frac{1}{\alpha} = \frac{1}{\alpha^2}$$ T<π

$$\frac{1}{V}E(z,\beta,V) = \frac{1}{\lambda^2}\frac{1}{\beta}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n^2} \equiv \frac{k_BT}{\lambda^2}f_2(z)$$

$$\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2 = -\frac{\partial}{\partial\beta}V\frac{2\pi m}{h^2}\frac{1}{\beta^2}f_2(z)$$

$$= 2V\frac{2\pi m}{h^2}\frac{1}{\beta^3}f_2(z) = 2V\frac{(k_BT)^2}{\lambda^2}f_2(z)$$

$$\langle\hat{H}\rangle = E(z,\beta,V) = V\frac{k_BT}{\lambda^2}f_2(z)$$

$$\boxed{\frac{\langle\hat{H}^2\rangle - \langle H\rangle^2}{\langle H\rangle^2} = 2\frac{\lambda^2}{V}\frac{1}{f_2(z)}}$$

IN 3D

$$\frac{E}{V} = \frac{3}{2}\frac{k_BT}{\lambda^3}f_{5/2}(z) \quad \left[= \frac{3}{2}P \;\; !\right]$$

$$-\frac{\partial E/V}{\partial\beta} = -\frac{\partial}{\partial\beta}\frac{3}{2}\left(\frac{2\pi m}{h^2}\right)^{3/2}\frac{1}{\beta^{5/2}}f_{5/2}(z) = \frac{15}{4}\frac{1}{\lambda^3\beta^2}f_{5/2}(z) = \frac{15}{4}\frac{(k_BT)^2}{\lambda^3}f_{5/2}(z)$$

$$\frac{\langle\hat{H}^2\rangle - \langle\hat{H}\rangle^2}{\langle\hat{H}\rangle^2} = \frac{15}{4}\frac{4}{9}\frac{\lambda^3}{V}\frac{1}{f_{5/2}(z)}$$

$$= \frac{5}{3}\frac{\lambda^3}{V}\frac{1}{f_{5/2}(z)}$$

③ $$z\frac{\partial \mathcal{N}}{\partial z}\Big|_{\beta V} = \frac{\sum_{N=0}^{\infty} N^2 z^N \mathrm{Tr}(e^{-\beta\hat{H}})}{\sum_{N=0}^{\infty} z^N \mathrm{Tr}(e^{-\beta\hat{H}})} - \left[\frac{\sum_{N=0}^{\infty} N z^N \mathrm{Tr}(e^{-\beta\hat{H}})}{\sum_{N=0}^{\infty} z^N \mathrm{Tr}(e^{-\beta\hat{H}})}\right]^2$$

$$= \langle N^2\rangle - \langle N\rangle^2$$

④
$$\frac{1}{V}\mathcal{N}(z,\beta,V) = \frac{1}{V}\sum_{\bar{p}} \frac{1}{\frac{e^{\beta p^2/2m}}{z}+1} =$$

$$= \frac{1}{h^2}\int d\bar{p}\, \frac{z e^{-\beta p^2/2m}}{1+z e^{-\beta p^2/2m}}$$

$$= \frac{2\pi}{h^2}\int_0^{\infty} dp\, p \sum_{n=1}^{\infty}(-1)^{n-1}\left(z e^{-\frac{\beta p^2}{2m}}\right)^n$$

$$= \frac{2\pi m}{h^2}\frac{1}{\beta}\int_0^{\infty} dx \sum_{n=1}^{\infty}(-1)^{n-1} z^n e^{-nx}$$

$$= \frac{1}{\lambda^2}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n} \equiv \frac{f_1(z)}{\lambda^2} = \frac{\ln(1+z)}{\lambda^2}$$

$$z\frac{\partial \mathcal{N}}{\partial z} = V\frac{f_0(z)}{\lambda^2}$$

$$f_0(z) = \sum_{n=1}^{\infty}(-1)^{n-1} z^n$$

$$f_0(z) = z\sum_{m=0}^{\infty}(-1)^m z^m$$

$$f_0(z) = \frac{z}{1+z}$$

$$\boxed{\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2} = \frac{\lambda^2}{V}\frac{f_0(z)}{[f_1(z)]^2}}$$

3D

$$\frac{n}{V} = \frac{f_{3/2}(z)}{\lambda^3}$$

$$z\frac{\partial n}{\partial z} = V\,\frac{f_{1/2}(z)}{\lambda^3}$$

$$\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2} = \frac{\lambda^3}{V}\,\frac{f_{1/2}(z)}{[f_{3/2}(z)]^2}$$

ESERCIZIO 2

① $g(E) = \frac{1}{A} \sum_{\bar{p}} \delta(E - \alpha p)$ $\qquad \alpha > 0$

$$= \frac{1}{A} \frac{A}{(2\pi\hbar)^2} \int d\bar{p}\, \delta(E-\alpha p) = \frac{2\pi}{(2\pi\hbar)^2} \int_0^\infty dp\, p\, \delta(E-\alpha p)$$

$$= \frac{1}{2\pi\hbar^2} \frac{p(E)}{\alpha} \qquad E > 0 \qquad p(E) = \frac{E}{\alpha}$$

$$\boxed{g(E) = \frac{E}{2\pi\hbar^2\alpha^2}\, \theta(E)}$$

②

$$\rho = \frac{1}{A} \sum_{\bar{p}} \frac{1}{\frac{e^{\beta\alpha p}}{z} - 1} = \int dE\, g(E) \frac{1}{\frac{e^{\beta E}}{z} - 1}$$

$$= \frac{1}{2\pi\hbar^2\alpha^2} \int_0^\infty dE \frac{E}{\frac{e^{\beta E}}{z} - 1} = \frac{1}{\ell^2} \int_0^\infty dx \frac{x}{\frac{e^x}{z} - 1}$$

$$\boxed{x = \beta E}$$

$$\rho = \frac{1}{\ell^2} \int_0^\infty dx \frac{x\, z e^{-x}}{1 - z e^{-x}}$$

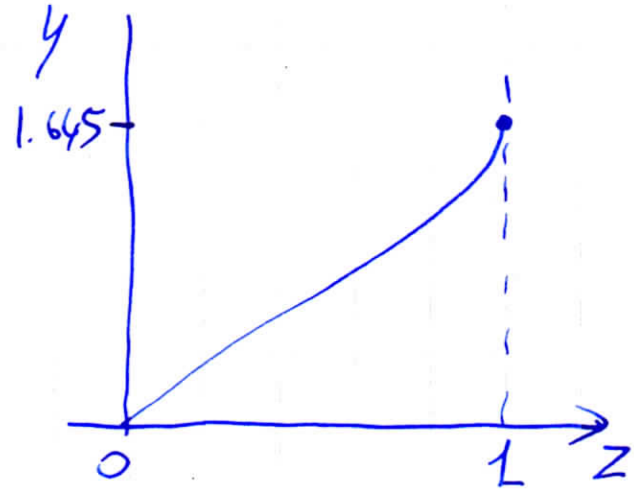
$$= \frac{1}{\ell^2} \int_0^\infty dx\, x \sum_{n=1}^\infty (z e^{-x})^n$$

$$= \frac{1}{\ell^2} \sum_{n=1}^\infty z^n \int_0^\infty dx\, x e^{-nx}$$

$$= \frac{1}{\ell^2} \sum_{n=1}^\infty z^n \left(-\frac{\partial}{\partial n}\right) \int_0^\infty dx\, e^{-nx} = \frac{1}{\ell^2} \sum_{n=1}^\infty \frac{z^n}{n^2}$$

$$\boxed{\rho = \frac{g_2(z)}{\ell^2}} \equiv y \qquad g_2(1) = \zeta(2) = 1.645$$

(3)



Per $\rho\ell^2 \gtrsim 1.645$ bisogna assumere che ci sia un condensato, ovvero

$$\rho = \frac{1}{A}\frac{z}{1-z} + \frac{g_2(1)}{\ell^2} \qquad \rho\ell^2 > \zeta(2)$$

(4) $(\ell^2\rho)_c = g_2(1) \Rightarrow \rho_c(T) = g_2(1)/\ell^2$

$$\rho = \rho_0 + \frac{g_2(1)}{\ell^2} \Rightarrow \frac{\rho_0}{\rho} = 1 - \frac{g_2(1)}{\rho\ell^2}$$

$$\boxed{\frac{\rho_0}{\rho} = 1 - \frac{\rho_c(T)}{\rho}}$$