II COMPITO – 21.12.18

ESERCIZIO 1

① P.B.C. $\bar{p} = \frac{h}{L}(\ell, m, n)$, $\ell, m, n \in \mathbb{Z}$

$$2 \cdot \frac{4\pi p_F^3/3}{(h/L)^3} = N \Rightarrow p_F^3 = \frac{h^3}{L^3} N \frac{3}{8\pi} = \hbar^3 \frac{N}{V} 8\pi^3 \frac{3}{8\pi} = \hbar^3 3\pi^2 \rho \;;\; \rho = N/V$$

$$\boxed{p_F = \hbar (3\pi^2 \rho)^{\frac{1}{3}}}$$

② $\tilde{\epsilon}(\bar{p}, s_z) = \sqrt{(pc)^2 + (m_o c^2)^2} - m_o c^2 = m_o c^2 \left[\sqrt{1 + (pc/m_o c^2)^2} - 1\right]$

(i) $m_o c^2 / p_F c \gg 1 \Rightarrow p \leq p_F :\ m_o c^2/pc \gg 1$, $pc/m_o c^2 \ll 1$

$$\tilde{\epsilon}(\bar{p}, s_z) \simeq m_o c^2 \left[1 + \frac{1}{2}\left(\frac{pc}{m_o c^2}\right)^2 - 1\right] = \frac{1}{2}\frac{(pc)^2}{m_o c^2}$$

$$\sum_{s_z, \bar{p}} \tilde{\epsilon}(\bar{p}, s_z) \simeq 2 \int \frac{d\bar{p}}{\frac{h^3}{V}} \cdot \frac{1}{2}\frac{(pc)^2}{m_o c^2} = \frac{4\pi V}{h^3} \int_0^{p_F} dp\, p^2 \frac{p^2}{m_o} = \frac{4\pi V}{h^3} \frac{p_F^5}{5 m_o}$$

$$E = \frac{V}{2\pi^2 \hbar^3} p_F^3 \frac{p_F^2}{5 m_o} = \frac{V}{2\pi^2 \hbar^3} \hbar^3 3\pi^2 \rho \frac{p_F^2}{5 m_o} = \frac{3}{5} N \frac{p_F^2}{2 m_o}$$

$$\boxed{E = N \frac{3}{5} \frac{p_F^2}{2 m_o} = N \frac{3}{5} \frac{\hbar^2 (3\pi^2 \rho)^{2/3}}{2 m_o}}$$

(ii) $m_o c^2 / p_F c \ll 1$

$$\tilde{\epsilon}(\bar{p}, s_z)/pc = \sqrt{1 + \left(\frac{m_o c^2}{pc}\right)^2} - \frac{m_o c^2}{pc} \simeq 1 + \frac{1}{2}\left(\frac{m_o c^2}{pc}\right)^2 - \frac{m_o c^2}{pc} \simeq 1$$

$$E = \sum_{s_z \bar{p}} pc = 2 \int \frac{d\bar{p}}{\frac{h^3}{V}} pc = \frac{8\pi V c}{h^3} \int_0^{p_F} dp\, p^2 p = \frac{8\pi V c}{h^3} \frac{p_F^4}{4} = \frac{2\pi V c}{h^3} \hbar^3 3\pi^2 \rho\, p_F$$

$$\boxed{E = \frac{3}{4} N p_F c}$$

③ (i)

$$E = \frac{3}{5} N \frac{\hbar^2}{2m_0}\left(3\pi^2 \frac{N}{V}\right)^{2/3} \; ; \; P = -\frac{\partial E}{\partial V} = \frac{2}{3}\frac{E}{V} \qquad \left(\frac{\partial}{\partial V} c V^{\alpha} = \frac{\alpha}{V} c V^{\alpha}\right)$$

(ii)

$$E = \frac{3}{4} N \hbar \left(3\pi^2 \frac{N}{V}\right)^{\frac{1}{3}} c \; ; \; P = -\frac{\partial E}{\partial V} = \frac{1}{3}\frac{E}{V}$$

$$\boxed{(i)\ P/(E/V) = \frac{2}{3} \; ; \; (ii)\ P/(E/V) = \frac{1}{3}}$$

④ $m_0 = 0 \Rightarrow p_F c \gg m_0 c^2 \Rightarrow \epsilon_F = p_F c = \hbar (3\pi^2 \rho)^{\frac{1}{3}} c$

$$\epsilon_F = 1.05 \times 10^{-27} (3\pi^2 \cdot 1)^{\frac{1}{3}} \cdot 3 \times 10^{10}\ \text{erg} = 9.75 \times 10^{-17}\ \text{erg}$$

$$\boxed{\epsilon_F(eV) = 9.75 \times 10^{-17} / (1.6 \times 10^{-12}) = 6.09 \times 10^{-5}}$$

ESERCIZIO 2

(1) $$\frac{\ln[\mathcal{Z}(z,V,T)]}{V} = \beta P = -\frac{1}{V}\sum_{\vec{p}} \ln\left[1 - z e^{-\beta\gamma_s p^s}\right] =$$

$$= -\frac{\Omega_D}{V}\int_0^\infty \frac{dp\, p^{D-1}}{\frac{h^D}{V}} \ln\left[1-ze^{-\beta\gamma_s p^s}\right] = -\frac{\Omega_D}{h^D}\int_0^\infty dp\,p^{D-1}\sum_{n=1}^\infty (-1)^{n-1}\frac{(-ze^{-\beta\gamma_s p^s})^n}{n}$$

$$= \frac{\Omega_D}{h^D}\sum_{n=1}^\infty (-1)^{n-1+n+1}\frac{z^n}{n}\int_0^\infty dp\,p^{D-1} e^{-n\beta\gamma_s p^s} \qquad \boxed{t = n\beta\gamma_s p^s \;(*)}$$

$$= \frac{\Omega_D}{h^D}\sum_{n=1}^\infty \frac{z^n}{n}\frac{1}{s}\int_0^\infty dt\, \frac{t^{D/s-1}}{(n\beta\gamma_s)^{D/s}} e^{-t} = \frac{\Omega_D}{h^D}\frac{\Gamma(D/s)}{s(\beta\gamma_s)^{D/s}}\sum_{n=1}^\infty \frac{z^n}{n^{D/s+1}}$$

$$= \frac{\Omega_D}{s h^D}\frac{\Gamma(D/s)}{(\beta\gamma_s)^{D/s}}\, g_{D/s+1}(z) \quad ; \quad g_\alpha(z) = \sum_{n=1}^\infty \frac{z^n}{n^\alpha}$$

$$\boxed{\beta P = \frac{\Omega_D}{s h^D}\frac{\Gamma(D/s)}{(\beta\gamma_s)^{D/s}}\, g_{\frac{D}{s}+1}(z)}\ \boxed{= \frac{1}{V}\ln\mathcal{Z}}$$

(2) $$\frac{E}{V} = -\frac{\partial}{\partial\beta}\frac{\ln\mathcal{Z}}{V}\bigg|_z = -\frac{\partial}{\partial\beta}(\beta P)\bigg|_z = \frac{D}{s}\frac{1}{\beta}\beta P$$

$$\boxed{E = \frac{D\,\Omega_D}{s^2 h^D}\frac{\Gamma(D/s)}{\beta(\beta\gamma_s)^{D/s}}\, g_{\frac{D}{s}+1}(z)}$$

(3) $$\boxed{\frac{P}{(E/V)} = s/D}$$

(*) $p = \left(t/n\beta\gamma_s\right)^{1/s}$, $dp = \frac{1}{s}\frac{p}{t}\,dt$, $dp\,p^{D-1} = \frac{1}{s}\left(\frac{t}{n\beta\gamma_s}\right)^{D/s}\frac{dt}{t}$

(4)

$$\rho = \frac{\langle N \rangle}{V} = \frac{1}{V} z \frac{\partial \ln \mathcal{Z}}{\partial z}\Big|_{\beta, V} = \frac{\Omega_D}{h^D s} \frac{\Gamma(D/s)}{(\beta \gamma_s)^{D/s}} \, g_{\frac{D}{s}}(z)$$

Per $z = 1$ otteniamo $\rho_c \propto g_{\frac{D}{s}}(1)$, a meno di un fattore finito.

Quindi $\lim_{z \to 1} g_{\frac{D}{s}}(z) = \infty$ implica $\frac{D}{s} \leq 1$.

La condensazione é assente per $\frac{D}{s} \leq 1$, ovvero per $D \leq s$

$$\boxed{\text{CONDENSAZIONE} \longrightarrow D > s}$$