II COMPITINO — 18.12.19 ESERCIZIO 1

① $\epsilon(\bar{p},\sigma) = \frac{p^2}{2m} - \mu_0 B\sigma$

$\min_{\sigma,\bar{p}} \epsilon(\bar{p},\sigma) = -\mu_0 B$, ottenuto per $\bar{p}=0$, $\sigma=1$; $\boxed{\mu < -\mu_0 B}$

② $M = \mu_0[\rho_+ - \rho_-] = \frac{\mu_0}{\lambda^3}\left[g_{3/2}(ze^{\beta\mu_0 B}) - g_{3/2}(ze^{-\beta\mu_0 B})\right] =$

$= \frac{\mu_0}{\lambda^3}\left[g_{3/2}(z\{1+\beta\mu_0 B\}) - g_{3/2}(z\{1-\beta\mu_0 B\})\right]$

D'altronde $g_{3/2}(z+\delta z_\sigma) = g_{3/2}(z) + g'_{3/2}(z)\delta z_\sigma = g_{1/2}(z)\frac{\delta z_\sigma}{z} = g_{1/2}(z)\frac{z\beta\mu_0 B\sigma}{z} = g_{1/2}(z)\beta\mu_0 B\sigma$

Ne segue che

$M \simeq \frac{\mu_0}{\lambda^3} g_{1/2}(z)\cdot 2\beta\mu_0 B = \frac{2\mu_0^2\beta}{\lambda^3} g_{1/2}(z)\, B$

③ $\chi = \left.\frac{\partial M}{\partial B}\right|_{z,T,B=0} = \frac{2\mu_0^2}{\lambda^3}\beta g_{1/2}(z) = 2^{5/2}\mu_0^2\left(\frac{\pi m}{h^2}\right)^{3/2} g_{1/2}(z)(k_B T)^{\frac{1}{2}}$

④ Per $z \to 1^-$, $g_{1/2}(z) = \sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}} \to \infty$. Dunque $\chi \to \infty$

ESERCIZIO 2

① Per $T \to 0^+$

$$n_+ = \vartheta[-\beta(\hbar v_F k - \mu)] = \vartheta[\beta(\mu - \hbar v_F k)] = 0, \ \forall\, k \geq 0.$$

Quindi $\mu < \hbar v_F k$, $\forall\, k \geq 0$, ovvero $\mu < 0$.

Similmente

$$n_- = \vartheta[-\beta(-\hbar v_F k - \mu)] = \vartheta[\beta(\mu + \hbar v_F k)] = 1, \ \forall\, k \geq 0;$$ da cui segue

$\mu + \hbar v_F k > 0$, $\forall\, k \geq 0$, ovvero $\mu > 0$.

Ne deduciamo $0 < \mu < 0$, ovvero

$$\boxed{\mu = 0}.$$

② $$1 - n_- = 1 - \frac{1}{e^{-\beta \hbar v_F k} + 1} = \frac{e^{-\beta \hbar v_F k}}{e^{-\beta \hbar v_F k} + 1} = \frac{1}{e^{\beta \hbar v_F k} + 1} = n_+$$

③ $$E(T) - E(0) = \sum_{S_z} \sum_{\vec{k}} \hbar v_F k \left[\left(n_+(k,T,0) - n_-(k,T,0)\right) - \left(n_+(k,0,0) - n_-(k,0,0)\right) \right] =$$

$$2 \sum_{\vec{k}} \hbar v_F k \left[n_+(k,T,0) - n_-(k,T,0) - 0 + 1 \right] = 2 \sum_{\vec{k}} \hbar v_F k \left[2 n_+(k,T,0) \right]$$

④

$$= 4 \int \frac{d\vec{k}}{\frac{(2\pi)^2}{L^2}} \frac{\hbar v_F k}{e^{\beta \hbar v_F k} + 1} = \frac{2}{\pi} A \int_0^\infty dk\, k \frac{\hbar v_F k}{e^{\beta \hbar v_F k} + 1} = \frac{2}{\pi} A \frac{(k_B T)^3}{(\hbar v_F)^2} \int_0^\infty dy \frac{y^2}{e^y + 1}$$

$$= \frac{2}{\pi} A\, \Gamma(3) f_3(1) \frac{(k_B T)^3}{(\hbar v_F)^2}$$

$y = \beta \hbar v_F k$

$s = my$

$$\boxed{C_V = \frac{1}{A} \frac{\partial [E(T) - E(0)]}{\partial T}\bigg|_{A, v_F} = \frac{12}{\pi} f_3(1) \left(\frac{k_B T}{\hbar v_F}\right)^2 k_B}$$

$$I = \int_0^\infty dy \frac{y^2}{e^y + 1} = \int_0^\infty dy \frac{y^2 e^{-y}}{1 + e^{-y}} = \int_0^\infty dy\, y^2 e^{-y} \sum_{n=0}^\infty (-e^{-y})^n = \sum_{n=0}^\infty (-1)^n \int_0^\infty dy\, y^2 e^{-(n+1)y} =$$

$$\sum_{m=1}^\infty \frac{(-1)^{m-1}}{m^3} \int_0^\infty ds\, s^2 e^{-s} = \Gamma(3) f_3(1) = 2 f_3(1) \quad ; \quad f_3(1) = \sum_{m=1}^\infty (-1)^{m-1} \frac{1}{m^3} = \frac{3}{4} \zeta(3).$$