ESERCIZIO 1

① $\varphi_p(x) = A \sin kx + B \cos kx$ $\boxed{p = \hbar k}$

$\varphi_p(0) = 0 = B$

$\varphi_p(L) = A \sin kL = 0$ $k = \frac{n\pi}{L}, n = 1,2,3,\dots$

$$\boxed{k = \frac{n\pi}{L}, \quad n = 1,2,3,\dots} \Rightarrow \boxed{k > 0}$$

②

$$g(E) = \frac{1}{L} \sum_k \delta(E - \hbar\alpha k)$$

$$= \frac{1}{\cancel{L}} \int_{\pi/L}^{\infty} dk \frac{1}{\pi/\cancel{L}} \delta(E - \hbar\alpha k)$$

$$= \frac{1}{\pi} \frac{1}{\hbar\alpha} \theta(E - \hbar\alpha \pi/L)$$

$$\boxed{g(E) = \frac{1}{\pi\hbar\alpha} \theta(E - \epsilon_0), \quad \epsilon_0 = \frac{\hbar\alpha\pi}{L}}$$

③

(1) $$\frac{1}{L} \ln Z = -\sum_k \frac{1}{L} \ln\left(1 - e^{-\beta \epsilon(k)} z\right)$$

$$= -\int dE \, g(E) \ln\left(1 - e^{-\beta E} z\right)$$

$$\frac{1}{L}\ln Z = -\frac{1}{\pi\hbar\alpha}\int dE\,\Theta(E-\epsilon_0)\ln(1-e^{-\beta E}z)$$

$$= -\frac{1}{\pi\hbar\alpha}\int_{\epsilon_0}^{\infty} dE\,\ln(1-e^{-\beta E}z)$$

$$= \frac{1}{\pi\hbar\alpha}\int_{\epsilon_0}^{\infty} dE\sum_{n=1}^{\infty}\frac{(e^{-\beta E}z)^n}{n}$$

$$= \frac{1}{\pi\hbar\alpha}\sum_{n=1}^{\infty}\frac{z^n}{n}\int_{\epsilon_0}^{\infty} dE\,e^{-\beta E n}$$

$$\boxed{y=\beta E n}$$

$$= \frac{1}{\pi\hbar\alpha}K_B T\sum_{n=1}^{\infty}\frac{z^n}{n^2}\int_{\beta n\epsilon_0}^{\infty} dy\,e^{-y}$$

$$= \frac{K_B T}{\pi\hbar\alpha}\sum_{n=1}^{\infty}\frac{z^n}{n^2}e^{-\beta\epsilon_0 n}$$

$$= \frac{K_B T}{\pi\hbar\alpha}\sum_{n=1}^{\infty}\frac{(z e^{-\beta\epsilon_0})^n}{n^2} = \frac{K_B T}{\pi\hbar\alpha}\,g_2(z e^{-\beta\epsilon_0})$$

$$\boxed{\frac{\ln Z}{L} = \frac{K_B T}{\pi\hbar\alpha}\,g_2(z e^{-\beta\epsilon_0})}$$

- Perché la (1) sia valida deve valere

$$z e^{-\beta\min(\epsilon(\alpha))} < 1 \Rightarrow z e^{-\beta\epsilon_0} < 1$$

• Poiché $\epsilon_0 = \hbar \alpha \frac{\pi}{L}$, nel limite termodinamico

$$\frac{\ln Z}{L} \longrightarrow \frac{k_B T}{\pi \hbar \alpha} g_2(z)$$

④ $$\rho = z \frac{\partial \ln Z / L}{\partial z} = \frac{k_B T}{\pi \hbar \alpha} g_1(z) \quad , \quad z < 1$$

Ma $$g_1(z) = \sum_{n=1}^{\infty} \frac{z}{n} = -\ln(1-z)$$

Quindi per $z \to 1$

$$\rho = -\frac{k_B T}{\pi \hbar \alpha} \ln(1-z) \longrightarrow \infty$$

e non c'è condensazione di Bose

ESERCIZIO 2

1)

$$\rho = \frac{N}{L} = \frac{1}{L}\sum_{S_z,p} = \frac{1}{\cancel{L}}\cdot 2\cdot\int_{-p_F}^{p_F}\frac{dp}{\frac{h}{\cancel{L}}} = \frac{4}{h}\int_0^{p_F} dp = 4\frac{p_F}{h}$$

$$= \frac{4}{2\pi}\frac{p_F}{\hbar} = \frac{2}{\pi}k_F \Rightarrow k_F = \frac{\pi}{2}\rho$$

$$\boxed{\epsilon_F = \frac{p_F^2}{2m}} \qquad \boxed{p_F = \hbar k_F = \hbar\frac{\pi}{2}\rho = \frac{h}{4}\rho} \qquad \boxed{\epsilon_F = \frac{h^2\rho^2}{32m}}$$

2)

$$g(E) = \frac{1}{L}\sum_{S_z}\sum_{p}\delta\left(E - \frac{p^2}{2m}\right)$$

$$= \frac{2}{\Delta}\int_{-\infty}^{+\infty}\frac{dp}{h/\cancel{\Delta}}\,\delta\left(E - \frac{p^2}{2m}\right) = \frac{4}{h}\int_0^{\infty} dp\,\delta\left(E - \frac{p^2}{2m}\right)$$

$$= \frac{4}{h}\frac{1}{\left|p/m\right|_{p=\sqrt{2mE}}} = \frac{4}{h}\frac{m}{\sqrt{2mE}}$$

$$= \frac{4}{2\pi}\frac{1}{\sqrt{2}}\frac{1}{\hbar}\sqrt{\frac{m}{E}} = \frac{\sqrt{2m}}{\pi\hbar}\frac{1}{\sqrt{E}}$$

$$\boxed{g(E) = \frac{\sqrt{2m}}{\pi\hbar}\frac{1}{\sqrt{E}}} = \frac{C'}{\sqrt{E}}$$

$$\int_0^{\epsilon_F} dE\, g(E) = \rho = C'\int_0^{\epsilon_F} dE\, E^{-1/2} = 2C' E^{\frac{1}{2}}\Big|_0^{\epsilon_F} = 2C'\epsilon_F^{1/2}$$

$$\boxed{C' = \rho/2\sqrt{\epsilon_F}}$$

3

$$\rho = \int_0^{\infty} dE \, \frac{g(E)}{e^{\beta(E-\mu)}+1} = \int_0^{\mu} dE \, g(E) + \frac{\pi^2}{6}(k_B T)^2 g'(\mu)$$

$$= C \left[\int_0^{\mu} dE \, E^{-\frac{1}{2}} + \frac{\pi^2}{6}(k_B T)^2 \frac{d}{dE} E^{-\frac{1}{2}} \Big|_{E=\mu} \right]$$

$$= C' \left[2\sqrt{\mu} + \frac{\pi^2}{6}(k_B T)^2 \left(-\frac{1}{2}\mu^{-3/2} \right) \right]$$

$$= C' \, 2\sqrt{\mu} \left[1 - \frac{\pi^2}{24} \frac{(k_B T)^2}{\mu^2} \right]$$

$$= \frac{\rho}{\cancel{2}\sqrt{E_F}} \cancel{2}\sqrt{\mu} \left[1 - \frac{\pi^2}{24} \left(\frac{k_B T}{\mu} \right)^2 \right]$$

$$\Rightarrow \sqrt{\frac{\mu}{E_F}} = \left[1 - \frac{\pi^2}{24} \left(\frac{k_B T}{\mu} \right)^2 \right]^{-1}$$

$$\mu = E_F \left[1 - \frac{\pi^2}{24} \left(\frac{k_B T}{\mu} \right)^2 \right]^{-2}$$

$$\simeq E_F \left[1 + \frac{\pi^2}{12} \left(\frac{k_B T}{\mu} \right)^2 \right]$$

$$\boxed{\mu \simeq E_F \left[1 + \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 + \cdots \right]}$$

4

$$\frac{E}{L} = \int_0^{\mu} dE\, g(E)E + \frac{\pi^2}{6}(K_B T)^2 \frac{d}{dE}(g(E)E)\Big|_{\mu}$$

$$\frac{E}{L} = C\left[\int_0^{\mu} dE\, E^{\frac{1}{2}} + \frac{\pi^2}{6}(K_B T)^2 \frac{d}{dE} E^{\frac{1}{2}}\Big|_{\mu}\right]$$

$$= C\left[\frac{2}{3}\mu^{3/2} + \frac{\pi^2}{6}(K_B T)^2 \frac{1}{2}\frac{1}{\mu^{1/2}}\right]$$

$$= C\,\frac{2}{3}\mu^{3/2}\left[1 + \frac{\pi^2}{8}(K_B T)^2\frac{1}{\mu^2}\right]$$

$$= \frac{\rho}{2\sqrt{E_F}}\frac{2}{3}\mu^{3/2}\left[1 + \frac{\pi^2}{8}\frac{(K_B T)^2}{\mu^2}\right]$$

$$= \frac{1}{3}\rho\,\epsilon_F\left(\frac{\mu}{\epsilon_F}\right)^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{K_B T}{\mu}\right)^2\right]$$

$$\frac{E}{L} = \frac{1}{3}\rho\,\epsilon_F\left[1 + \frac{\pi^2}{12}\left(\frac{K_B T}{\mu}\right)^2\right]^{3/2}\left[1 + \frac{\pi^2}{8}\left(\frac{K_B T}{\mu}\right)^2\right]$$

$$\simeq \frac{1}{3}\rho\,\epsilon_F\left[1 + \pi^2\left(\frac{1}{8} + \frac{3}{2}\frac{1}{12}\right)\left(\frac{K_B T}{\mu}\right)^2\right]$$

$$\simeq \frac{1}{3}\rho\,\epsilon_F\left[1 + \frac{\pi^2}{4}\left(\frac{K_B T}{\epsilon_F}\right)^2\right]$$

$$\boxed{\frac{E}{L} = \frac{1}{3}\rho\,\epsilon_F\left[1 + \frac{\pi^2}{4}\left(\frac{K_B T}{\epsilon_F}\right)^2\right]}$$

$$C_V = \frac{\partial (E/L)}{\partial T}\Big|_{\rho} = \frac{\pi^2}{6}\rho\,\cancel{\epsilon_F}\frac{K_B T}{\epsilon_F^{\cancel{2}}}K_B$$

- Ricordiamo che $g(\epsilon_F) = \rho/2\epsilon_F$. Quindi

$$\boxed{C_V = \frac{\pi^2}{6} \rho K_B \left(\frac{K_B T}{\epsilon_F}\right) = \frac{\pi^2}{3} g(\epsilon_F)(K_B T) K_B}$$