I COMPITO – ESERCIZIO 1

① $\Sigma(E,V,N_1,N_2) = \dfrac{1}{N_1!\,N_2!\,h^{3N_1}h^{3N_2}} \cdot \int\limits_{H\leq E} dp_1 \dots dp_N \cdot \int d\bar{P}_1 \dots d\bar{P}_{N_2} \cdot$

$$\cdot \int d\bar{r}_1 \dots \int d\bar{r}_N \cdot \int d\bar{R}_1 \dots \int d\bar{R}_N =$$

$$= \frac{(\sqrt{2m_1})^{3N_1}(2m_2)^{3N_2}}{N_1!\,N_2!\,h^{3(N_1+N_2)}} \cdot \int\limits_{\sum_{i=1}^{3(N_1+N_2)} x_i^2 \leq E = R^2} dx_1\,dx_2 \dots dx_{3(N_1+N_2)} \cdot V^{N_1+N_2}$$

Evidentemente c'è bisogno del volume Ω_{3N} di una ipersfera di raggio R in $3(N_1+N_2) \equiv 3N$ dimensioni:

$$\Omega_{3N} = \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2}+1)} E^{3N/2}$$

Quindi $\Sigma(E,V,N_1,N_2) = \dfrac{((2m_1)^{3/2}V)^{N_1}((2m_2)^{3/2}V)^{N_2}}{N_1!\,N_2!\,h^{3N_1}\,h^{3N_2}} \; \dfrac{\pi^{3/2 N_1}\;\pi^{3/2 N_2}}{\Gamma(\frac{3N}{2}+1)} E^{\frac{3}{2}(N_1+N_2)}$

$$= \frac{1}{N_1!}\left(\frac{2\pi m_1 E}{h^2}\right)^{3/2 N_1} \cdot V^{N_1} \cdot \frac{1}{N_2!}\left(\frac{2\pi m_2 E}{h^2}\right)^{3/2 N_2} V^{N_2} \; \frac{1}{\Gamma(\frac{3}{2}N+1)}$$

Per l'entropia, utilizzando l'approssimazione di Stirling, $\Gamma(n+1) \simeq (n/e)^n$, $n \gg 1$, otteniamo

$$S(E,V,N_1,N_2) = k_B\left[N_1 \ln\left(\frac{eV}{N_1}\left(\frac{2\pi m_1 E}{h^2}\right)^{3/2}\right) + N_2 \ln\left(\frac{eV}{N_2}\left(\frac{2\pi m_2 E}{h^2}\right)^{3/2}\right)\right.$$

$$\left. + \frac{3}{2}N \ln\frac{2e}{3N}\right].$$

② $$\frac{1}{T} = \left.\frac{\partial S}{\partial E}\right|_{V,N_1,N_2} = k_B \frac{\partial}{\partial E}\left[N_1 \ln E^{3/2} + N_2 \ln E^{3/2}\right] = \frac{3}{2}\frac{k_B(N_1+N_2)}{E}$$

$$\Rightarrow \quad T = \frac{E}{\frac{3}{2}k_B(N_1+N_2)}, \qquad E = \frac{3}{2}(N_1+N_2)k_B T .$$

② $$P = T\left.\frac{\partial S}{\partial V}\right|_{E,N_1,N_2} = k_B T \frac{\partial}{\partial V}\left[N_1 \ln V + N_2 \ln V + \text{cost}\right]$$

$$= k_B T \frac{N_1+N_2}{V}$$

③ $$\mu_1 = -T\left.\frac{\partial S}{\partial N_1}\right|_{E,V,N_2} = -k_B T \frac{\partial}{\partial N_1}\left[N_1 \ln\left(\frac{eV}{N_1}\left(\frac{2\pi m_1 E}{h^2}\right)^{3/2}\right) + \frac{3}{2}N \ln\left(\frac{2e}{3N}\right) + \text{cost}\right]$$

$$\mu_1 = -k_B T\left[\ln\left[\frac{eV}{N_1}\left(\frac{2\pi m_1 E}{h^2}\right)^{3/2}\right] - \cancel{1} + \frac{3}{2}\ln\left(\frac{2e}{3N}\right) - \cancel{\frac{3}{2}}\right]$$

$$\mu_1 = -k_B T \ln\left[\frac{V}{N_1}\left(\frac{2\pi m_1 E \cdot 2}{h^2\, 3N}\right)^{3/2}\right] = k_B T \ln\left[\frac{N_1}{V}\left(\frac{h^2}{\frac{4\pi m_1 E}{3N}}\right)^{3/2}\right]$$

Utilizzando l'espressione di $E(T)$ ottenuta al punto ②, troviamo

$$\mu_1 = k_B T \ln\left[\rho_1\left(\frac{h^2}{2\pi m_1 k_B T}\right)^{3/2}\right] = k_B T\left(\ln \rho_1 \lambda_1^3\right)$$

Evidentemente

$$\mu_2 = k_B T \ln\left(\rho_2 \lambda_2^3\right)$$

e

$$\lambda_\alpha^2 = \frac{h^2}{2\pi m_\alpha k_B T} \qquad \left[= \frac{h^2}{\frac{4\pi m_\alpha E}{3N}}\right].$$

ESERCIZIO 2

① $\mathcal{H} = \sum_{i=1}^{N} h^{(1)}(i)$

$h^{(1)} = \frac{p^2}{2m} + v \ln\left[\left(\frac{r}{R}\right)^\alpha\right] \qquad 0 < r < R$

$Q_N(V,T) = \frac{1}{N!} q^N$

$q = \frac{1}{h^3}\int d\bar{r} \int d\bar{p}\, e^{-\beta h} = \frac{1}{h^3}\int d\bar{p}\, e^{-\frac{\beta p^2}{2m}} \int_{0<r<R} d\bar{r}\, e^{-\beta v \ln\left(\frac{r}{R}\right)^\alpha}$

$= \frac{1}{\lambda^3} 4\pi \int_0^R dr\, r^2\, e^{-\ln\left(\frac{r}{R}\right)^{\beta v \alpha}}$

$\beta v \alpha \equiv \gamma < 3$

$= \frac{4\pi}{\lambda^3}\int_0^R dr\, r^2 \left(\frac{R}{r}\right)^\gamma = \frac{4\pi}{\lambda^3} R^\gamma \int_0^R dr\, r^{2-\gamma}$

$= \frac{4\pi}{\lambda^3}\frac{R^\gamma}{3-\gamma} R^{3-\gamma} = \frac{1}{\lambda^3}\frac{4\pi}{3-\gamma} R^3$

$$\boxed{Q_N(V,T) = \frac{1}{N!}\left[\frac{4\pi}{3-\gamma}\frac{R^3}{\lambda^3}\right]^N \simeq \left[\frac{e}{N}\frac{4\pi}{3-\gamma}\frac{R^3}{\lambda^3}\right]^N}$$

$$\boxed{A(N,V,T) = -N k_B T \ln\left[\frac{e}{N}\frac{V}{\lambda^3}\frac{3}{3-\gamma}\right] \quad V = \frac{4\pi}{3}R^3}$$

$A(N,V,T) = -N k_B T \ln\left[\frac{eV}{N\lambda^3}\frac{3}{3-\beta v \alpha}\right]$

② $$E = \frac{\partial \beta A}{\partial \beta} = -N \frac{\partial}{\partial \beta} \ln\left[\frac{eV}{N\lambda^3} \frac{3}{3-\beta U \alpha}\right]$$

$$= -N \frac{\partial}{\partial \beta}\left[\ln\left[\beta^{-3/2}\right] - \ln\left[3-\beta U \alpha\right]\right]$$

$$\boxed{E = N k_B T\left[\frac{3}{2} - \frac{\beta U \alpha}{3-\beta U \alpha}\right]} \qquad \lambda \sim \beta^{1/2}$$

③ $$S = \frac{E-A}{T} = N k_B\left[\frac{3}{2} - \frac{\beta U \alpha}{3-\beta U \alpha} + \ln\left(\frac{eV}{\lambda^3 N} \frac{3}{3-\beta U \alpha}\right)\right]$$

④ $$\rho(r) = N \langle \delta(\bar{r}-\bar{r}_1)\rangle = N \frac{\int d\bar{r}_1\, \delta(\bar{r}-\bar{r}_1)\, e^{-\beta U(\bar{r}_1)}}{\int d\bar{r}_1\, e^{-\beta U(\bar{r}_1)}}$$

$$= \frac{N e^{-\beta U(r)}}{\frac{4\pi}{3-\gamma} R^3} = N \frac{(R/r)^\gamma}{\frac{4\pi}{3-\gamma} R^3} = \frac{N}{\frac{4\pi R^3}{3}} \frac{3-\gamma}{3}\left(\frac{R}{r}\right)^\gamma$$

$$\rho_0 = \frac{N}{V} \Rightarrow \boxed{\rho(r) = \rho_0 \frac{3-\gamma}{3}\left(\frac{R}{r}\right)^\gamma}$$

$$\frac{\rho(r)}{\rho(R)} = \left(\frac{R}{r}\right)^\gamma \equiv \tilde{\rho}\left(\frac{R}{r}\right) \equiv \tilde{\rho}(x) \qquad x = r/R$$