COMPITO DEL 21/12/17 – ESERCIZIO I

① $$Q(V,T) = \sum_{\{n_{\bar p\sigma}\}} e^{-\beta \sum_{\bar p\sigma} \epsilon_{\bar p\sigma} n_{\bar p\sigma}} = \sum_{\{n_{\bar p\sigma}\}} \prod_{\bar p\sigma} e^{-\beta \epsilon_{\bar p\sigma} n_{\bar p\sigma}} =$$

$$= \prod_{\bar p\sigma} \sum_{n_{\bar p\sigma}=0}^{1} e^{-\beta \epsilon_{\bar p\sigma} n_{\bar p\sigma}} = \prod_{\bar p\sigma} \left(1+e^{-\beta\epsilon_{\bar p\sigma}}\right) = \prod_{\bar p\sigma}\left(1+e^{-\beta c p}\right)$$

$$A(V,T) = -k_B T \ln Q(V,T) = -k_B T \sum_{\bar p\sigma} \ln\left(1+e^{-\beta c p}\right) =$$

$$-2k_BT\sum_{\bar p}\ln\left(1+e^{-\beta c p}\right) = -2k_BT\sum_{\bar p}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{e^{-n\beta c p}}{n}$$

$$= -2k_BT\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}\sum_{\bar p} e^{-n\beta c p} = -2k_BT\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\int \frac{d\bar p}{\frac{h^3}{V}} e^{-n\beta c p}$$

$$= -\frac{2k_BTV}{h^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\, 4\pi\int_0^{\infty} dp\, p^2 e^{-n\beta c p} \qquad \boxed{t = n\beta c p}$$

$$= -\frac{8\pi k_BTV}{h^3}\frac{1}{(\beta c)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}\int_0^{\infty} dt\, t^2 e^{-t} = -V\frac{8\pi (k_BT)^4}{(hc)^3}\Gamma(3)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4}$$

$$\boxed{\frac{A(V,T)}{V} = -\frac{2}{\pi^2}\frac{(k_BT)^4}{(\hbar c)^3} f_4(1), \qquad f_4(z) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n^4}}$$

② $$\sum_\sigma \langle n_{\bar p\sigma}\rangle = -\frac{\partial \ln Q}{\partial \beta c p} = \frac{\partial}{\partial \beta c p}\left[-2\sum_q \ln\left(1+e^{-\beta c q}\right)\right] =$$

$$= \frac{2e^{-\beta c p}}{1+e^{-\beta c p}} \qquad \boxed{\sum_\sigma \langle n_{\bar p\sigma}\rangle = \frac{2}{e^{\beta c p}+1}}\ ^{*}$$

③ $$\frac{E(V,T)}{V} = \frac{\partial}{\partial\beta}\left(\frac{\beta A}{V}\right) = \frac{\partial}{\partial\beta}\left(-\frac{2}{\pi^2}\frac{1}{(\hbar c)^3}\frac{1}{\beta^3}f_4(1)\right) = \frac{6}{\pi^2}\frac{1}{(\hbar c)^3}\frac{1}{\beta^4}f_4(1)$$

$$\boxed{\frac{E(V,T)}{V} = \frac{6}{\pi^2}\frac{(k_B T)^4}{(\hbar c)^3}f_4(1)}$$

④ $$C_V = \frac{\partial}{\partial T}\left(\frac{E}{V}\right) = \frac{24}{\pi^2}\frac{(k_B T)^3}{(\hbar c)^3}f_4(1)\cdot k_B$$

$$\boxed{C_V = \frac{24}{\pi^2}\frac{(k_B T)^3}{(\hbar c)^3}f_4(1)\,k_B}$$

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* $$Q = \sum_{\{n_{\bar p\sigma}\}} e^{-\beta\sum_{\bar p\sigma}\epsilon_{\bar p\sigma}n_{\bar p\sigma}} = \sum_{\{n_{\bar p\sigma}\}} e^{-\beta c\sum_{\bar p} p\sum_\sigma n_{\bar p\sigma}}$$

$$-\frac{\partial \ln Q}{\partial \beta c q} = \frac{1}{Q}\sum_{\{n_{\bar p\sigma}\}}\left(\sum_\sigma n_{\bar q\sigma}\right)e^{-\beta c\sum_{\bar p} p\sum_\sigma n_{\bar p\sigma}} = \left\langle \sum_\sigma n_{\bar q\sigma}\right\rangle$$

ESERCIZIO II

① $\ln \mathcal{Z} = -\sum_{\vec p}' \ln\left(1 - e^{-\beta \epsilon_{\vec p}} z\right)$

$$\frac{1}{A}\ln \mathcal{Z} = -\frac{1}{A}\int \frac{d\vec p}{\frac{h^2}{A}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(-e^{-\beta\epsilon_p} z\right)^n = \frac{1}{h^2} 2\pi \int_0^\infty dp\, p \sum_{n=1}^{\infty} \frac{z^n}{n} e^{-n\beta\gamma p^s}$$

$$= \sum_{n=1}^{\infty} \frac{2\pi}{h^2}\frac{z^n}{n}\int_0^\infty dp\, p\, e^{-n\beta\gamma p^s} = \sum_{n=1}^{\infty} \frac{2\pi}{h^2}\frac{z^n}{n}\frac{1}{s(n\beta\gamma)^{2/s}}\int_0^\infty dt\, t^{2/s-1} e^{-t}$$

$$\boxed{\begin{aligned} &n\beta\gamma p^s = t \\ &p = \left(\frac{t}{n\beta\gamma}\right)^{\frac{1}{s}} \\ &dp = \frac{dt}{t}\frac{1}{s}\left(\frac{t}{n\beta\gamma}\right)^{\frac{1}{s}} \end{aligned}}$$

$$= \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s(\beta\gamma)^{2/s}}\sum_{n=1}^{\infty}\frac{z^n}{n^{2/s+1}} = \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s}\left(\frac{k_B T}{\gamma}\right)^{2/s} g_{2/s+1}(z)$$

NOTA: trascuriamo il termine anomalo $-\frac{1}{A}\ln(1-z)$

$$\boxed{\frac{1}{A}\ln Z = \beta P = \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s}\left(\frac{k_B T}{\gamma}\right)^{2/s} g_{2/s+1}(z)}$$

② $\frac{E}{A} = -\frac{\partial}{\partial\beta}\frac{1}{A}\ln\mathcal{Z}(z,A,\beta)\Big|_z = -\frac{\partial}{\partial\beta} X\cdot\beta^{-2/s} = \frac{2}{s}X\beta^{-2/s-1} = \frac{2}{s}X\beta^{-2/s}\cdot\beta^{-1}$

$$= \frac{2}{s}\frac{1}{\beta}\frac{1}{A}\ln\mathcal{Z} = \frac{2}{s}\frac{1}{\beta}\beta P = \frac{2}{s}P$$

$$\boxed{\left(\frac{E}{A}\right)\Big/P = \frac{2}{s}}$$

③ $\rho_n = z\frac{\partial}{\partial z}\frac{1}{A}\ln\mathcal{Z} = \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s}\left(\frac{k_B T}{\gamma}\right)^{2/s} z\, g'_{2/s+1}(z) = \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s}\left(\frac{k_B T}{\gamma}\right)^{2/s} g_{2/s}(z)$

Trascuriamo il termine anomalo $\frac{1}{A}\frac{z}{1-z}$

$$\boxed{\rho_n(z,t) = \frac{2\pi}{h^2}\frac{\Gamma(2/s)}{s}\left(\frac{k_B T}{\gamma}\right)^{2/s} g_{2/s}(z)}$$

(4) $\rho_n = \frac{2\pi}{h^2} \frac{\Gamma(2/s)}{s} \left(\frac{k_B T}{\gamma}\right)^{2/s} g_{2/s}(z)$

$g_{2/s}(1) = \zeta\left(\frac{2}{s}\right)$, $\zeta(x)$ essendo la funzione di Riemann

$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, $\zeta(x) < \infty$ per $x > 1$

Quindi per $z=1$ $\rho_n < \infty$ per $\frac{2}{s} > 1$, ovvero per $s < 2$

$$\boxed{\text{CONDENSAZIONE PER } s < 2}$$