COMPITO DEL 19-12-14

ESERCIZIO 1.

① $$g(E) = \frac{1}{V}\sum_{\vec{p}, S_z}{}' \delta\left(E - \frac{p^2}{2m}\right) = \frac{g_s}{V}\int \frac{d\vec{p}}{h^4/V}\,\delta\left(E-\frac{p^2}{2m}\right) =$$

$$g_s \frac{2\pi^2}{h^4}\int_0^\infty dp\, p^3\, \delta\left(E - \frac{p^2}{2m}\right) \;;\; x = p^2/2m,\; p = \sqrt{2mx},\; dp = \sqrt{\frac{m}{2x}}\,dx \;,\; \boxed{g_s = 2S+1 = 4}$$

$$g(E) = \frac{2\pi^2}{h^4} g_s \int_0^\infty dx \sqrt{\frac{m}{2x}}\,(2mx)^{3/2}\,\delta(E-x) = \frac{4\pi^2 m^2}{h^2} g_s \int_0^\infty dx\, x\, \delta(E-x)$$

$$\boxed{g(E) = \left(\frac{2\pi m}{h^2}\right)^2 g_s E,\; E \geq 0}$$

② $$\frac{\langle N\rangle}{V} \equiv \rho = \int_0^\infty dE \frac{g(E)}{e^{\beta(E-\mu)}+1} = g_s\left(\frac{2\pi m}{h^2}\right)^2 \int_0^\infty dE \frac{E}{e^{\beta(E-\mu)}+1} = \frac{g_s}{\lambda^4}\int_0^\infty dt \frac{t}{\frac{e^t}{z}+1}$$

$$= \frac{g_s}{\lambda^4}\int_0^\infty dt \frac{z t e^{-t}}{1+z e^{-t}} \equiv \frac{g_s}{\lambda^4} f(z)$$

$$z<1 \;;\; f(z) = \int_0^\infty dt\, z t e^{-t} \sum_{n=0}^\infty (-1)^n z^n e^{-nt} = \sum_{n=1}^\infty (-1)^{n-1} z^n \int_0^\infty dt\, t e^{-nt}$$

$$\int_0^\infty dt\, t e^{-nt} = -t\frac{e^{-nt}}{n}\Big|_0^\infty + \int_0^\infty dt \frac{e^{-nt}}{n} = -\frac{e^{-nt}}{n^2}\Big|_0^\infty = \frac{1}{n^2}$$

$$f(z) = \sum_{n=1}^\infty (-1)^{n-1}\frac{z^n}{n^2} \equiv f_2(z)$$

$$\boxed{z<1 \;;\; \rho = \frac{g_s}{\lambda^4} f_2(z)}$$

③ $$\beta P = \frac{\ln Z}{V} = \frac{g_s}{V}\sum_{\vec{p}} \ln\left[1 + z e^{-\beta \epsilon_{\vec{p}}}\right] \;;\qquad \epsilon_{\vec{p}} = p^2/2m$$

$$\beta P = \int_0^\infty dE\, g(E) \ln\left[1 + z e^{-\beta E}\right] = \frac{g_s}{\lambda^4}\int_0^\infty dt\, t \ln\left[1 + z e^{-t}\right] \equiv \frac{g_s}{\lambda^4}\tilde{f}(z)$$

$$\tilde{f}(z) = \int_0^\infty dt\, t \sum_{n=1}^\infty (-1)^{n-1} z^n \frac{e^{-nt}}{n} = \sum_{n=1}^\infty (-1)^{n-1}\frac{z^n}{n}\int_0^\infty dt\, t e^{-nt} = \sum_{n=1}^\infty (-1)^{n-1}\frac{z^n}{n^3} \equiv f_3(z)$$

④ $\rho\lambda^4 \to 0$

$$y = \frac{\rho\lambda^4}{g_s} = z - \frac{z^2}{4} + \frac{z^3}{9} + O(z^4) \qquad (1)$$

$$\beta P = \frac{g_s}{\lambda^4}\left[z - \frac{z^2}{8} + \frac{z^3}{27} + O(z^4)\right] \qquad (2)$$

$$z = y + a y^2 + b y^3 + \cdots$$

$$(1) \Rightarrow \quad y = y + a y^2 + b y^3 - \frac{1}{4}(y^2 + 2a y^3) + \frac{y^3}{9}$$

$$a - \frac{1}{4} = 0 \Rightarrow a = \frac{1}{4}; \quad b - \frac{2a}{4} + \frac{1}{9} = b - \frac{1}{8} + \frac{1}{9} = 0 \Rightarrow b = \frac{1}{72}$$

$$z = y + \frac{1}{4}y^2 + \frac{y^3}{72}$$

$$(2) \Rightarrow \quad \beta P = \frac{g_s}{\lambda^4}\left[y + \frac{1}{4}y^2 + \frac{y^3}{72} - \frac{1}{8}\left(y^2 + \frac{y^3}{2}\right) + \frac{y^3}{27}\right] = \frac{g_s}{\lambda^4}\left[y + \frac{y^2}{8} - \frac{5y^3}{432}\right]$$

$$\boxed{\beta P = \rho\left[1 + \frac{1}{8}\frac{\rho\lambda^4}{g_s} - \frac{5}{432}\left(\frac{\rho\lambda^4}{g_s}\right)^2 + \cdots\right]}$$

ESERCIZIO 2.

① $\rho = \frac{1}{V}\sum_{\vec{p}} \frac{1}{\frac{e^{\beta\alpha p}}{z}-1} = \frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \frac{1}{\frac{e^{\beta\alpha p}}{z}-1} = 4\pi\left(\frac{k_BT}{h\alpha}\right)^3 \int_0^\infty dt\, t^2 \frac{ze^{-t}}{1-ze^{-t}}$

$\equiv \frac{1}{\ell^3} g(z)$; $t = \beta\alpha p$, $\frac{1}{\ell^3} = 8\pi\left(\frac{k_BT}{h\alpha}\right)^3$

$g(z) = \frac{1}{2}\int_0^\infty dt\, t^2 z e^{-t} \sum_{n=0}^\infty z^n e^{-nt} = \frac{1}{2}\sum_{n=1}^\infty z^n \int_0^\infty dt\, t^2 e^{-nt} = \sum_{n=1}^\infty \frac{z^n}{n^3} \equiv g_3(z)$

$\rho = \frac{1}{\ell^3} g_3(z)$

③ $\beta P = \frac{1}{V}\sum_{\vec{p}}(-)\ln\left[1-ze^{-\beta\alpha p}\right] = -\frac{4\pi}{h^3}\int_0^\infty dp\, p^2 \ln\left[1-ze^{-\beta\alpha p}\right] =$

$\frac{1}{\ell^3}\cdot\frac{1}{2}\int_0^\infty dt\, t^2 \ln\left[1-ze^{-t}\right] = \frac{1}{\ell^3}\frac{1}{2}\int_0^\infty dt\, t^2 \sum_{n=1}^\infty \frac{z^n e^{-nt}}{n} = \frac{1}{\ell^3}\sum_{n=1}^\infty \frac{z^n}{n}\int_0^\infty dt\, t^2 \frac{e^{-nt}}{2} =$

$\frac{1}{\ell^3}\sum_{n=1}^\infty \frac{z^n}{n^4} \equiv \frac{1}{\ell^3} g_4(z)$

③ $(\rho\ell^3)_c = g_3(1) < \infty \Rightarrow (\rho\ell^3) > g_3(1)$, $\rho_0 > 0$!

④ $\rho = \rho_0 + \frac{g_3(1)}{\ell^3}$, $\rho > \rho_c(T) = \frac{g_3(1)}{\ell^3(T)}$; $\frac{\rho_0}{\rho} = 1 - \frac{g_3(1)}{\ell^3(T)\rho} = 1 - \frac{\rho_c(T)}{\rho}$

$1 - \frac{\rho_c(T)}{\rho}$

$\rho = \rho_0 + \frac{g_3(1)}{\ell^3}$, $\ell^3(T) > \ell^3(T_c) = \frac{g_3(1)}{\rho}$; $\frac{\rho_0}{\rho} = 1 - \frac{g_3(1)}{\ell^3(T)\rho} = 1 - \frac{\ell^3(T_c)}{\ell^3(T)} =$

$1 - \left(\frac{T}{T_c}\right)^3$