Condensed Matter Physics II. – A.A. 2011-2012, June 20, 2012

(time 3 hours)

Solve the following two exercises, each has a maximum score of 18 for a total of 36. A score between 33 e 36 corresponds to 30 cum laude, between 30 e 32 is renormalized to 30 (the maximum official score, without laude).

NOTE:

- Give all details which help in understanding the proposed solution. Answers which only contain the final result or not enough detail will be judged insufficient and discarded;
- If you are requested to give evaluation/estimates, do so using 3 significant figures.

Exercise 1: *Excitation in a linear Debye chain.*

Consider the harmonic vibrations (phonons) in an infinite linear chain of equispaced atoms, with lattice parameter a, and springs of constant G connecting each atom to its nearest neighbors.

1. Let $u_+(q,n,t) = \epsilon \exp[i(qna - \omega(q)t)]$ and $u_-(q,n,t) = u_+^*(q,n,t)$ be the 2N independent solutions (normal modes) of the dynamical problem, with q, n, t respectively a wavevector in the FBZ, a lattice position and time. We have in mind a chain of length $L = Na$, with PBC. Let's resort to Debye approximation and replace $\omega(q)$ with the linear behavior valid for $|q| \ll \pi/a$. Give explicitely $\omega(q)$ in such approximation for both positive and negative values of q.

2. Let's consider a superposition of the normal modes with coefficients

$$a_-(q) = a_+(q) = \frac{l\,a}{5L}\frac{1}{1+(ql)^2}$$

and (i) $l \gg a/\pi$. Calculate

$$u(n,t) = \sum_{\sigma=\pm,q} a_\sigma(q) u_\sigma(q,n,t).$$

We remark that due to the condition (i) above, the integral over q can be approximated extending it to all q-space, i.e., over the q-range $[-\infty, \infty]$.

3. Are there atoms displaced from the equilibrium positions at $t = 0$.

4. Calculate the speed of each atom at $t = 0$.

5. Which atoms are displaced from equilibrium at $t = ma/c$, with $ma \gg l$ and c the sound velocity: please, answer by giving also a qualitative sketch of the displacements along the chain!

6. Give a qualitative sketch of the velocity of the atoms along the chain, at $t = ma/c$, with $ma \gg l$ and c the sound velocity: please provide a detailed motivation of the sketch.

Note:

$$\int_0^\infty dq \frac{\cos[qs]}{1+(ql)^2} = \frac{\pi}{2l} e^{-|s|/l}$$

Esercizio 2 *Pauli susceptibility in 2D at $T \neq 0$*

Consider a non interacting electron gas in 2 dimensions, in a uniform magnetic field $\mathbf{H} = H\hat{z}$. The energy levels of the $\sigma = \pm$ (i.e., up or down) spin electrons are $e_\sigma(\mathbf{k}) = \hbar^2 k^2/(2m) + \sigma\mu_B H$. We shall denote with n the areal number density and A the area of the sample.

1. Knowing that total density of states at $H = 0$,
$$g_0(E) = \frac{2}{A}\sum_{\mathbf{k}} \delta(E - \frac{\hbar^2 k^2}{2m}) = \frac{n}{\epsilon_F}\theta(E),$$
express
$$g_\sigma(E) = \frac{1}{A}\sum_{\mathbf{k}} \delta(E - e_\sigma(\mathbf{k}))$$
in terms of $g_0(E) = (n/\epsilon_F)\theta(E)$. Here, $\theta(x)$ is the Heaviside step function.

2. Assuming that the chemical potential μ is given calculate the density of σ electrons at givel T, H, μ from the relation
$$n_\sigma = \int dE \, \frac{g_\sigma(E)}{exp[\beta(E-\mu)] + 1},$$
where the range of the energy integration is determined by the range of the density of states $g_\sigma(E)$.

3. Using the result found above, calculate the magnetization density $M(T, H, \mu) = -\mu_B(n_+ - n_-)$.

4. Imposing the constraint $n_+(T, H, \mu) + n_-(T, H, \mu) = n$ show that $\mu(T, H) = \mu(T, -H)$, i.e., μ is an even function of H.

5. Assuming that $\mu(T, H) = \mu_0(T) + \alpha H^2, \quad H \to 0$, expand M to linear order in H as $M(T, H, \mu) = M(T, \mu_0) + \chi(T, \mu_0)H$, obtaining the explicit expression of $\chi(T, \mu_0)$.

6. Knowing that $\mu_0 = \epsilon_F + K_B T \log[1 - exp(-\beta\epsilon_F)]$, express $\chi(T, \mu_0)$ in terms of n, ϵ_F, β. This expression is valid at arbitrary temperature.

   Note: $\int dx \, [(exp(x) + 1)]^{-1} = -\log[(1 + exp(-x)]$.