II COMPITO 07.06.17

ESERCIZIO 1

① $$\epsilon(\bar{k}) = g\mu_B H + 2S \sum_{\bar{R}}{}' J(\bar{R}) \sin^2\left(\frac{\bar{k}\cdot\bar{R}}{2}\right)$$

$$= g\mu_B H + 2S \sum_{\bar{R},\, p.v.}{}' J \sin^2\left(\frac{\bar{k}\cdot\bar{R}}{2}\right)$$

$$= g\mu_B H + 2SJ \sum_{\bar{R}=(\pm a,0),(0,\pm a)}{}' \sin^2\left(\frac{\bar{k}\cdot\bar{R}}{2}\right)$$

$$\epsilon(H,\bar{k}) = g\mu_B H + 2SJ\left[2\sin^2\left(\frac{k_x a}{2}\right) + 2\sin^2\left(\frac{k_y a}{2}\right)\right]$$

$ka \ll 1$

$$\sin^2\left(\frac{k_x a}{2}\right) \simeq \frac{k_x^2 a^2}{4}$$

$$\epsilon(H,\bar{k}) \simeq g\mu_B H + 2SJ\left[2\frac{(k_x^2+k_y^2)a^2}{4}\right]$$

$$= g\mu_B H + SJk^2a^2$$

$$\boxed{\epsilon(H,\bar{k}) = g\mu_B H + \frac{J}{2}k^2a^2}$$

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(3)

$$\boxed{\epsilon_D(H,\bar{k}) = g\mu_B H + \frac{Ja^2}{2}k^2 \qquad k < k_D}$$

$$\frac{\pi k_D^2}{\left(\frac{2\pi}{L}\right)^2} = \frac{A k_D^2}{4\pi} = N \Rightarrow \boxed{k_D^2 = 4\pi\frac{N}{A} = \frac{4\pi}{a^2}}$$

$$g_D(E) = \frac{1}{A}\sum_{\bar{k}} \delta(E - \epsilon_D(\bar{k},H))$$

$$= \frac{1}{A}\sum_{\bar{k}} \delta\left(E - g\mu_B H - \frac{J}{2}a^2k^2\right) \qquad \boxed{k \leq k_D}$$

$$g_D(E) = \frac{1}{\cancel{A}}\int_{k\leq k_D} \frac{dk\, k\, 2\pi}{\frac{(2\pi)^2}{\cancel{A}}} \delta\left(E - g\mu_B H - \frac{J}{2}a^2k^2\right)$$

$$= \frac{1}{2\pi}\int_0^{k_D} dk\, k\, \delta\left(E - g\mu_B H - \frac{Ja^2}{2}k^2\right)$$

$$= \frac{2}{4\pi a^2}\int_0^{(k_D a)^2} dy\, \delta(E - g\mu_B H - Jy) \qquad y = \frac{k^2a^2}{2}$$

$$g_D(E) = \frac{1}{2\pi a^2 J}\,\theta(E - g\mu_B H) \qquad \boxed{\begin{matrix} E \leq g\mu_B H + \frac{J}{2}k_D^2a^2 \\ E \geq g\mu_B H \end{matrix}}$$

$$= 0 \qquad \boxed{E > g\mu_B H + \frac{J}{2}k_D^2a^2}$$

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$$(33.30)\quad M(T) = M(0)\left[1 - \frac{1}{NS}\sum_{k} n(\bar{k})\right] \qquad M(0) = \frac{NS}{A}$$

Nel caso presente $n(\bar{k}) = \dfrac{1}{e^{\beta \epsilon_D(\bar{k},H)} - 1} \equiv n(H,\bar{k})$

$$M(H,T) = M(0)\left[1 - \frac{A}{NS}\frac{1}{A}\sum_{\bar{k}} \frac{1}{e^{\beta\epsilon_D(\bar{k},H)} - 1}\right]$$

$$M(H,T) = M(0)\left[1 - \frac{A}{NS}\int_{g\mu_B H}^{g\mu_B H + Ja^2k_D^2/2} dE\, \frac{g_D(E)}{e^{\beta E} - 1}\right]$$

5 $$M(H,T) = \frac{NS}{A}\left[1 - \frac{A}{NS}\frac{1}{2\pi a^2 J}\int_{g\mu_B H}^{g\mu_B H + Ja^2k_D^2/2} \frac{dE}{e^{\beta E} - 1}\right]$$

$$M(H,T) = \frac{NS}{A}\left[1 - \frac{A}{NS}\frac{KT}{2\pi a^2 J}\ln\frac{1 - e^{-\beta(g\mu_B H + Jk_D^2a^2/2)}}{1 - e^{-\beta g\mu_B H}}\right]$$

$$M(H,T) = M(0)\left[1 - \frac{K_B T}{\pi J}\ln\left[\frac{1 - e^{-\beta(g\mu_B H + Jk_D^2a^2/2)}}{1 - e^{-g\mu_B H}}\right]\right]$$

(6) $\epsilon_D = k_B T_D = \frac{J}{2} a^2 k_D^2 = \frac{J}{2} a^2 \frac{4\pi}{a^2} = \frac{4\pi J}{2} = 2\pi J$

$\epsilon_D = \frac{1}{2}\, eV \quad , \quad T_D = \frac{0.5\, eV}{8.62 \cdot 10^{-5}\, eV\, K^{-1}} = 0.580 \cdot 10^4\, K$

$$\boxed{T_D = 0.580 \times 10^4\, K}$$

$$M(H,T) = M(0)\left[1 - \frac{2T}{T_D} \ln\left(\frac{1 - e^{-\beta g \mu_B H} \times e^{-\frac{T_D}{T}}}{1 - e^{-\beta g \mu_B H}}\right)\right]$$

$T \ll T_D$ fissato $\qquad g\mu_B H \ll k_B T$

$$M(H,T) \simeq M(0)\left[1 - \frac{2T}{T_D} \ln \frac{1}{\beta g \mu_B H}\right]$$

$$M(H,T) = 0 \Rightarrow \ln \frac{1}{\beta g \mu_B H} = \frac{T_D}{2T}$$

$$\ln \beta g \mu_B H = -\frac{T_D}{2T}$$

$$\boxed{g\mu_B H_0 = k_B T\, e^{-\frac{T_D}{2T}}} \boxed{= g\mu_B H_0(T)}$$

Quando H raggiunge, da valori più grandi, il valore $H_0(T)$ la magnetizzazione si annulla.

ESERCIZIO 2.

① $\psi(\bar r_1,\bar r_2) = \varphi_0(\bar r) = \sum'_{\epsilon(\bar k)>\epsilon_F} g(\bar k) e^{i\bar k \bar r} \qquad \bar r = \bar r_1 - \bar r_2$

$$\left[-\frac{\hbar^2}{2m}(\nabla_1^2+\nabla_2^2) + U(\bar r_1-\bar r_2) - E\right] \sum'_{\bar k} g(\bar k) e^{i\bar k(\bar r_1-\bar r_2)} = 0$$

Moltiplicando a sinistra per $\frac{e^{-i\bar q(\bar r_1-\bar r_2)}}{V}$ ed integrando in $\bar r_1-\bar r_2$ si ha

$$\sum'_{\bar k} g(\bar k)\left\{\left[2\epsilon(\bar q) - E\right]\delta_{\bar k\bar q} + \langle \bar q|U|k\rangle\right\} = 0$$

Ovvero

② $$\left(2\epsilon(\bar q) - E\right) g(\bar q) - \frac{U_1}{V}\sum_{\substack{\bar k \\ \epsilon_F \le \epsilon(\bar k) \le \epsilon_F + \hbar\omega_D}} \hat q\cdot\hat k\, g(\bar k) = 0$$

③ Singlet

$$\varphi_0(-\bar r) = \sum'_{\bar k} g(k) e^{-i\bar k\cdot \bar r} = \sum'_{\bar Q} g(-\bar Q) e^{i\bar Q\cdot\bar r}$$

$$= \varphi_0(\bar r) = \sum'_{\bar Q} g(\bar Q)\, e^{i\bar Q r} \Rightarrow g(-\bar Q) = g(\bar Q)$$

$$\sum'_{k} \equiv \sum_{\substack{\bar k \\ \epsilon(\bar k)>\epsilon_F}} \quad ; \quad \epsilon(k) = \frac{\hbar^2 k^2}{2m}$$

Triplet

$$\varphi_0(-\bar{r}) = \sum_{\bar{k}}{}' g(\bar{k}) e^{-i\bar{k}\cdot\bar{r}} = \sum_{\bar{Q}}{}' g(-\bar{Q}) e^{i\bar{Q}\cdot r}$$

$$= -\varphi_0(\bar{r}) = -\sum_{\bar{Q}}{}' g(\bar{Q}) e^{i\bar{Q}\cdot\bar{r}} \Rightarrow g(-\bar{Q}) = -g(\bar{Q})$$

④ For the singlet

$$\hat{p}\cdot\sum_{\bar{k}}{}' \hat{k} g(\bar{k}) = \hat{p} \sum_{\bar{Q}}{}' -\hat{Q} g(-\bar{Q}) = -\hat{p} \sum_{\bar{Q}}{}' \hat{Q} g(\bar{Q})$$

$$\Rightarrow \sum_{\bar{k}}{}' \hat{k} g(\bar{k}) = 0$$

The minimum energy for the singlet is

$$E = 2\epsilon_F$$

since the potential does not contribute

(5) Let us consider in eq. (2) the second term on the r.h.s.

$$-\frac{U_1}{V}\sum_{\vec{k}}{}' \hat{q}\cdot\hat{k}\, g(\vec{k})$$

and expand $g(\vec{k})$

$$g(\vec{k}) = \sum_{l'm'} g_{l'm'}(k)\, Y_{l'm'}(\vartheta',\varphi')$$

using also

$$\hat{q}\cdot\hat{k} = \frac{4\pi}{3}\sum_{m=-1}^{1} Y^*_{1,m}(\vartheta',\varphi')\, Y_{1,m}(\vartheta,\varphi).$$

We get

$$-U_1 \int' \frac{dk\,k^2}{(2\pi)^3}\int d\Omega' \,\frac{4\pi}{3}\sum_{m=-1}^{1} Y^*_{1m}(\vartheta'\varphi')\, Y_{1m}(\vartheta,\varphi)$$

$$\times \sum_{l'm'} g_{l'm'}(k)\, Y_{l'm'}(\vartheta',\varphi')$$

$$= -U_1\,\frac{4\pi}{3}\int' \frac{dk\,k^2}{(2\pi)^3}\sum_{m=-1}^{1} Y_{1m}(\vartheta,\varphi)\sum_{l'm'} g_{l'm'}(k)\,\delta_{l',1}\,\delta_{mm'}$$

$$= -U_1\,\frac{4\pi}{3}\int' \frac{dk\,k^2}{(2\pi)^3}\sum_{m=-1}^{1} g_{1m}(k)\, Y_{1,m}(\vartheta,\varphi)$$

This yields

$$[2\epsilon(q)-E]\sum_{l'm'} g_{l'm'}(q)\, Y_{l'm'}(\vartheta,\varphi) =$$

$$v_1 \frac{4\pi}{3}\int^1 \frac{d\kappa\,\kappa^2}{(2\pi)^3}\sum_{m=-1}^{1} g_{1m}(\kappa)\, Y_{1m}(\vartheta,\varphi)$$

If we multiply by $Y^*_{\tilde{l}\tilde{m}}(\vartheta,\varphi)$ and integrate over Ω we get

$$[2\epsilon(q)-E]\, g_{\tilde{l}\tilde{m}}(q) = \frac{4\pi}{3} v_1 \int^1 \frac{d\kappa\,\kappa^2}{(2\pi)^3}\,\delta_{\tilde{l},1}\, g_{1\tilde{m}}(\kappa)$$

Thus for $\tilde{l} \neq 1$

$$[2\epsilon(q)-E]\, g_{\tilde{l},\tilde{m}}(q) = 0 \quad \Rightarrow \quad E \geq 2\epsilon_F$$

and for $\tilde{l}=1$

$$g_{1\tilde{m}}(q) = \frac{4\pi v_1/3}{2\epsilon(q)-E}\int^1 \frac{d\kappa\,\kappa^2}{(2\pi)^3}\, g_{1\tilde{m}}(\kappa)$$

and

$$\int^1 \frac{dq\, q^2}{(2\pi)^3}\, g_{1\tilde{m}}(q) = \frac{4\pi}{3} v_1 \int^1 \frac{dq\, q^2}{(2\pi)^3}\, \frac{1}{2\epsilon(q)-E} \cdot \int^1 \frac{d\kappa\,\kappa^2}{(2\pi)^3}\, g_{1\tilde{m}}(\kappa)$$

which implies

$$1 = \frac{4\pi}{3} v_1 \int^1 \frac{dq\, q^2}{(2\pi)^3}\,\frac{1}{2\epsilon(q)-E}$$

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(5) $$1 = \frac{V_1}{3}\frac{1}{V}\sum_{\bar{q}}{}' \frac{1}{2\epsilon(q) - E}$$

$$= \frac{V_1}{3}\int d\epsilon \frac{g_0(\epsilon)/2}{2\epsilon - E}$$ $\epsilon_F \leq \epsilon \leq \epsilon_F + \hbar\omega_D$

As $\hbar\omega_D \ll \epsilon_F$ we get

$$1 \simeq \frac{V_1}{3}\frac{g_0(\epsilon_F)}{2}\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D} \frac{d\epsilon}{2\epsilon - E}$$ $\boxed{n_0 \equiv g(\epsilon_F)/2}$

$$= \frac{V_1}{3}\frac{g_0(\epsilon_F)}{2}\frac{1}{2}\ln\frac{2\epsilon_F + 2\hbar\omega_D - E}{2\epsilon_F - E}$$

$$e^{\frac{6}{V_1 n_0}} = 1 + \frac{2\hbar\omega_D}{2\epsilon_F - E}$$ $\mathcal{E} = 2\epsilon_F - E$

$$\frac{\mathcal{E}}{2\hbar\omega_D} = \frac{1}{e^{6/V_1 n_0} - 1}$$ $V_1 n_0 \ll 1$

$$\boxed{\mathcal{E} \simeq 2\hbar\omega_D\, e^{-6/V_1 n_0}}$$ $E < 2\epsilon_F$