Condensed Matter Physics II. – A.A. 2023-2024, June 05, 2024

(time 3 hours)

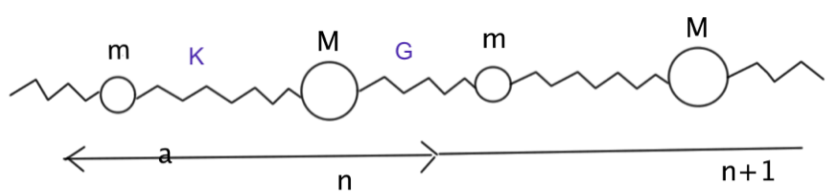
Solve the following two exercises, each has a maximum score of 18 for a total of 36. A score between 33 e 36 corresponds to 30 cum laude, between 30 e 32 is renormalized to 30 (the maximum official score, without laude).

NOTE:

- Give all details which help in understanding the proposed solution. Answers which only contain the final result or not enough detail will be judged insufficient and discarded;
- If you are requested to give evaluation/estimates, do so using 3 significant figures.

Exercise 1: *Harmonic linear chain with two atoms per unit cell*

Consider a linear harmonic chain with 2 atoms (with masses m and M) in a unit cell of length a. The atoms are connected to the nearest neighbours by springs with harmonic constant K and G.



1. Write the total potential energy of the linear chain when the atoms are displaced from the equilibrium positions: the displacement of atoms from equilibrium in the n-th cell will be denoted by $u_1(n)$ and $u_2(n)$, respectively for the atoms with mass m and M.
2. Write down the equation of motion for the atoms in the n-th cell
3. Making the ansatz that the displacements have a wavelike behaviour $u_1(n,t) = \epsilon_1 exp(i(qna - \omega(q)t)$ and $u_2(n,t) = \epsilon_2 exp(i(qna - \omega(q)t)$, obtain the allowed values of frequencies $\omega_\pm(q)$, i.e. the dispersion of the two phonon branches.
4. Give the values of the obtained 2 phonon branches at $q = 0$ and at $q = \pi/a$.
5. Show that for q in $[0, \pi/a]$ one of the two branches is an increasing function of q and the other a decreasing function of q. It may help studying the derivative of $\omega_\pm(q)^2$ with respect to q. Give a qualitative sketch of the dispersion of the two branches paying attention to the form of the dispersion around $q = 0$ and around $q = \pi/a$.
6. Provide a quantitative sketch of the 2 branches for q in $[0, \pi/a]$ when $m = M/2$ and $K = G/2$. Plot $\omega_\pm(q)/\tilde{\omega}$, with

$$\tilde{\omega} = \frac{3}{\sqrt{2}}\sqrt{\frac{G}{M}}$$

.

Exercise 2: *Spin paramagnetism in a 2D electron gas*

1. Calculate (or just write it down if you know it) the energy density of state $g(E)$ for spin unpolarized electrons and express it in terms of the areal density $n = N/A$ and the Fermi energy ϵ_F. What are the dimensions of $g(E)$? Sketch $g(E)$, being carefull to the allowed energy range.

2. A magnetic field $H = H\hat{z}$ is applied to the electrons, which move in the plane (x, y). Write the interaction energy with the field for an electron with spin projection S_z (in units of $\hbar$). Take the electron g-factor equal to 2 and express the results in terms of the Bohr magneton μ_B. [Note, as the electrons move in 2 dimension and the field is in plane, one has to consider only the coupling of the spin with the field, i.e., there are no orbital terms.]

3. Calculate the density of states of spin up and spin down electrons, $g_+(E)$ and $g_-(E)$ and sketch them. [Be careful to specify the allowed energy ranges for $g_+(E)$ and $g_-(E)$].

4. Derive for the present case and for arbitrary values of H and T the expression of the magnetization density induced by the field. You can write it in term of the densities n_+ and n_- of up and down electrons, which have simple expressions in terms of the Fermi function $f(E)$ and the densities of states $g_+(E)$ and $g_-(E)$. Proceed as much as possible to obtain an expression in terms of n, ϵ_F, μ, H, T.

5. Neglecting the dependence of the chemical potential μ on temperature and magnetic field and assuming that $\mu_B \ll \epsilon_F$ and $K_B T \ll \epsilon_F$, first plot on the same graph $f(E - \mu_B H)$ and $f(E + \mu_B H)$; calculate then the magnetization density to leading order in H.

6. Give the expression of the resulting magnetic susceptibility and comment its relation with the known 3-dimensional result.