ESERCIZIO 1 - COMPITO 5.6.24

①

$$U = \frac{1}{2}\sum_n \left\{ K\left[u_1(n) - u_2(n)\right]^2 + G\left[u_2(n) - u_1(n+1)\right]^2 \right\}$$

②

$$m\ddot{u}_1(\ell) = -\frac{\partial U}{\partial u_1(\ell)} = -K\left[u_1(\ell) - u_2(\ell)\right] - G\left[u_1(\ell) - u_2(\ell-1)\right]$$

$$M\ddot{u}_2(\ell) = -\frac{\partial U}{\partial u_2(\ell)} = -K\left[u_2(\ell) - u_1(\ell)\right] - G\left[u_2(\ell) - u_1(\ell+1)\right]$$

③

$$u_\alpha(\ell, t) = \epsilon_\alpha e^{i[qa\ell - \omega_q t]}$$

$$\Rightarrow \quad -m\omega_q^2 \epsilon_1 e^{i[qa\ell - \omega_q t]} = -K\left[\epsilon_1 - \epsilon_2\right] e^{i[qa\ell - \omega_q t]} - G\left[\epsilon_1 - \epsilon_2 e^{-iqa}\right] e^{i[qa\ell - \omega_q t]}$$

$$-M\omega_q^2 \epsilon_2 e^{i[qa\ell - \omega_q t]} = -K\left[\epsilon_2 - \epsilon_1\right] e^{i[qa\ell - \omega_q t]} - G\left[\epsilon_2 - \epsilon_1 e^{iqa}\right] e^{i[qa\ell - \omega_q t]}$$

$\Rightarrow$

$$m\omega_q^2\epsilon_1 - (K+G)\epsilon_1 + (K+Ge^{-iqa})\epsilon_2 = 0$$

$$(K+Ge^{iqa})\epsilon_1 + M\omega_q^2\epsilon_2 - (K+G)\epsilon_2 = 0$$

$$\Rightarrow \begin{pmatrix} m\omega_q^2 - (K+G) & (K+Ge^{-iqa}) \\ (K+Ge^{iqa} & M\omega_q^2 - (K+G) \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0$$

$$\Rightarrow [m\omega_q^2 - (K+G)][M\omega_q^2 - (K+G)] - |K+Ge^{iqa}|^2 = 0$$

$$mM\omega_q^4 - (K+G)(m+M)\omega_q^2 + (K+G)^2 - |K+Ge^{iqa}|^2 = 0$$

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \omega_\pm^2(q) = \frac{1}{2mM}\Big\{(m+M)(K+G)$$

$$\pm \sqrt{(m+M)^2(K+G)^2 - 4mM\left[|K+G|^2 - |K+Ge^{iqa}|^2\right]}$$

Note that

$$(K+G)^2 - (K+Ge^{iqa})(K+Ge^{-iqa})$$

$$= \cancel{K^2} + \cancel{G^2} + 2KG - \cancel{K^2} - \cancel{G^2} - KG(e^{iqa} + e^{-iqa}$$

$$= 2KG[1-\cos qa] = 4KG\sin^2\left(\frac{qa}{2}\right)$$

Thus

$$\omega_\pm^2(q) = \frac{1}{2mM}\Big\{(m+M)(K+G)$$

$$\pm\sqrt{(m+M)^2(K+G)^2 - 16mMKG\sin^2\left(\frac{qa}{2}\right)}\Big\}$$

(4)

$$(1)\ \omega_\pm^2(0) = \frac{1}{2mM}\Big\{(m+M)(K+G) \pm (m+M)(K+G)\Big\}$$

$$\omega_+^2(0) = \frac{(m+M)(K+G)}{mM}$$

$$\omega_-^2(0) = 0$$

(2)

$$\omega_\pm^2\left(\frac{\pi}{a}\right) = \frac{1}{2mM}\Big\{(m+M)(K+G) \pm$$

$$\sqrt{(m+M)^2(K+G)^2 - 16mMKG}\Big\}$$

$$\omega_\pm^2\left(\frac{\pi}{a}\right) = \frac{(m+M)(K+G)}{2mM}\left\{1 \pm \sqrt{1 - \frac{16mMKG}{(m+M)^2(K+G)^2}}\right\}$$

$$\omega_+\left(\frac{\pi}{a}\right) > \omega_-\left(\frac{\pi}{a}\right)$$

5

$$\frac{d\,\omega_\pm^2(q)}{dq} = \pm\frac{1}{2}\frac{1}{2mM}\frac{1}{\sqrt{(m+M)^2(K+G)^2-16mMKG\sin^2\left(\frac{qa}{2}\right)}}$$

$$\times\left[-16mMKG\cdot 2\sin\frac{qa}{2}\cos\frac{qa}{2}\,\frac{a}{2}\right]$$

$$\frac{d\,\omega_\pm^2(q)}{dq} = \pm\frac{1}{4mM}\frac{1}{\sqrt{\cdots}}\left[-8mMKG\sin(qa)\cdot a\right]$$

$$= \mp\frac{2KGa\sin qa}{\sqrt{\cdots}}$$

$0<q<\frac{\pi}{a}$ $\qquad \sin qa > 0$

$$\frac{d\,\omega_+^2(q)}{dq} < 0 \qquad\qquad \frac{d\,\omega_-^2(q)}{dq} > 0$$

$qa \ll 1)$

$$\omega_\pm^2(q) = \frac{(m+M)(K+G)}{2mM}\left\{1\pm\sqrt{1-\frac{16mMKG\sin^2\left(\frac{qa}{2}\right)}{(m+M)^2(K+G)^2}}\right\}$$

$$\omega_\pm^2(q) \simeq \frac{(m+M)(K+G)}{2mM}\left\{1\pm 1\mp\frac{4mMKGa^2q^2}{(m+M)^2(K+G)}\right\}$$

$$\left.\begin{aligned} \omega_-^2(q) &\sim \gamma a^2 q^2 \\ \omega_+^2(q) &\sim 2 - \gamma a^2 q^2 \end{aligned}\right\} \quad \gamma > 0$$

$$\omega_-(q) \sim c\, q \quad , \quad \omega_+(q) \sim \sqrt{2 - \gamma a^2 q^2}$$

$$\epsilon = \frac{\pi}{2} - \frac{qa}{2} \qquad 0 < \epsilon \ll 1$$

$$\sin^2\left(\frac{qa}{2}\right) = \sin^2\left(\frac{\pi}{2} - \epsilon\right) = \cos^2(\epsilon) \simeq 1 - \epsilon^2$$

$$\sqrt{1 - 4\gamma \sin^2\left(\frac{qa}{2}\right)} \simeq \sqrt{1 - 4\gamma\left[1 - \left(\frac{\pi}{2} - \frac{qa}{2}\right)^2\right]}$$

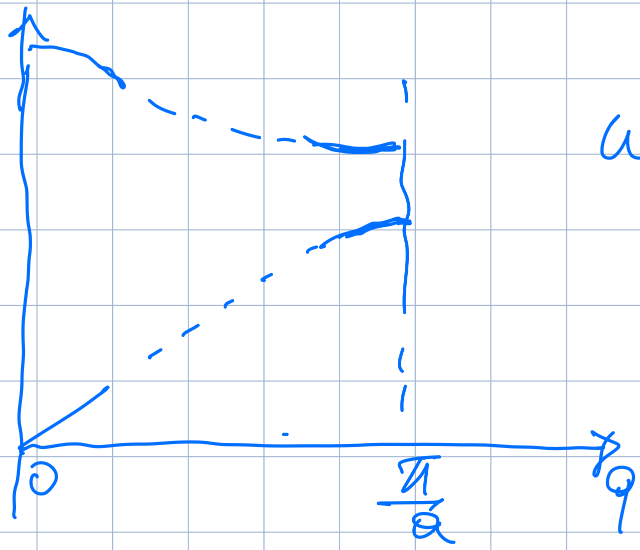
$$= \sqrt{1 - 4\gamma + 4\gamma\left(\frac{\pi}{2} - \frac{qa}{2}\right)^2} \simeq$$

$$\sqrt{1 - 4\gamma}\left\{1 + \frac{2\gamma}{1 - 4\gamma}\left[\frac{\pi}{2} - \left(\frac{qa}{2}\right)^2\right]\right\}$$

$$\omega_\pm^2(q) \propto 1 \pm \left\{\sqrt{1 - 4\gamma} + \frac{2\gamma}{\sqrt{1 - 4\gamma}}\left(\frac{\pi}{2} - \frac{qa}{2}\right)^2\right\}$$

$$= 1 \pm \sqrt{1 - 4\gamma} \pm \frac{2\gamma}{\sqrt{1 - 4\gamma}}\left(\frac{\pi}{2} - \frac{qa}{2}\right)^2$$

$$= C_\pm \pm \Delta\left(\frac{\pi}{2} - \frac{qa}{2}\right)^2 \qquad \Delta > 0$$



$$\omega_\pm(q) = \sqrt{C_\pm} \pm \frac{\Delta}{\sqrt{C_\pm}}\left(\frac{\pi}{a} - \frac{qa}{2}\right)^2$$

$$4\gamma = \frac{16 m M K G}{(m+M)(K+G)} < 1$$

$$\frac{4mM}{(M+m)^2} = \frac{(M+m)^2 - (M-m)^2}{(M+m)^2} \leq 1$$

$$\frac{4KG}{(K+G)^2} \leq 1$$

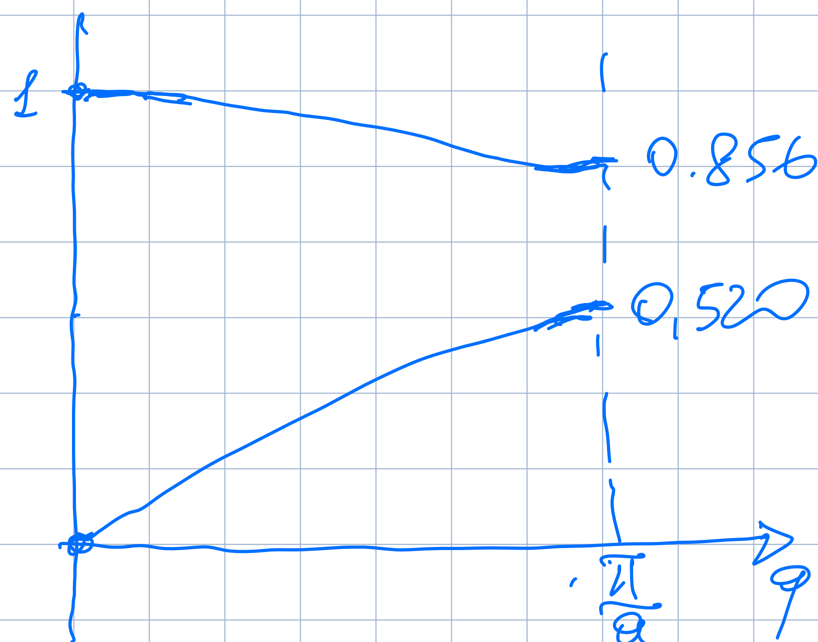
⑥ $m = \frac{M}{2}$, $K = \frac{G}{2}$

$$\omega_\pm^2(q) = \frac{\frac{3}{2}M\,\frac{3}{2}G}{M^2}\left\{1 \pm \sqrt{1 - \frac{4M^2G^2}{\left(\frac{3}{2}\right)^4 M^2G^2}\sin^2\left(\frac{qa}{2}\right)}\right\}$$

$$= \frac{9}{4}\frac{G}{M}\left\{1 \pm \sqrt{1 - \frac{64}{81}\sin^2\left(\frac{qa}{2}\right)}\right\}$$

$$= \tilde{\omega}^2\,\frac{1}{2}\left\{1 \pm \sqrt{1 - \frac{64}{81}\sin^2\frac{qa}{2}}\right\}$$

$$\frac{\omega_\pm(q)}{\tilde{\omega}} = \frac{1}{\sqrt{2}}\left\{1 \pm \sqrt{1 - \frac{64}{81}\sin^2\left(\frac{qa}{2}\right)}\right\}^{\frac{1}{2}}$$



ESERCIZIO 12

① $$g(E) = \frac{n}{\epsilon_F}\,\theta(E) = \frac{2}{A}\sum_{\vec{k}} \delta(E-\epsilon(\vec{k}))$$

② $$U = -\vec{\mu}\cdot\vec{H} = \mu_B\, g\, S_z H = \mu_B H \sigma$$

$\sigma = \pm 1$ per $S_z = \pm\frac{1}{2}$

$$\boxed{\epsilon(k) = \frac{\hbar^2 k^2}{2m}}$$

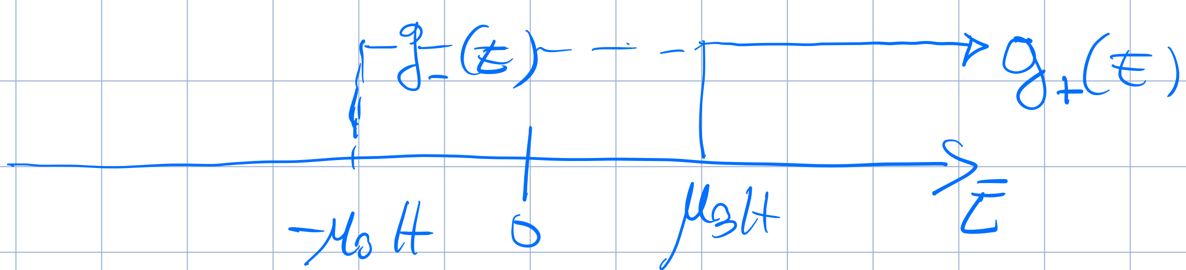
③ $$\epsilon(k,\sigma) = \epsilon(k) + \mu_B H \sigma$$

$$g_\sigma = \frac{1}{A}\sum_{\vec{k}} \delta(E-\epsilon(\vec{k},\sigma))$$

$$= \frac{1}{A}\sum_{k} \delta(E-\mu_B H\sigma-\epsilon(k))$$

$$= \frac{1}{2}\, g(E-\mu_B H\sigma)$$

$$g_\sigma(E) = \frac{n}{2\epsilon_F}\,\theta(E-\mu_B H\sigma)$$



(4) $$n_\sigma = \int dE\, g_\sigma(E)\, f(E)$$

$$f(E) = \frac{1}{e^{\beta(E-\mu)}+1}$$

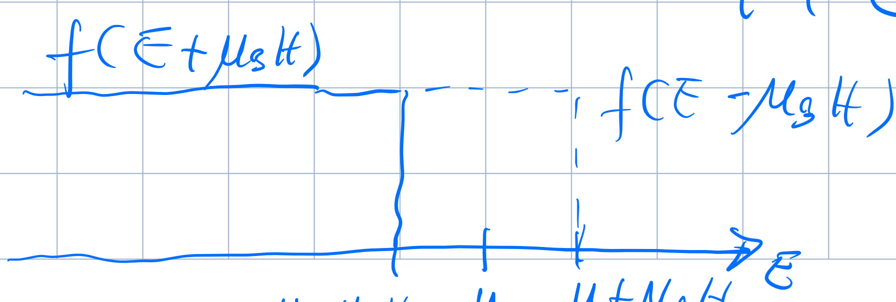
$$M = -\mu_B\left[n_+ - n_-\right] = -\mu_B\left[\int dE\, g_+(E) f(E) - \int dE\, g_-(E)\, f(E)\right] =$$

$$= -\frac{\mu_B n_0}{2\epsilon_F}\left\{\int_{\mu_B H}^{\infty} dE\, f(E) - \int_{-\mu_B H}^{\infty} dE\, f(E)\right\}$$

$$= \frac{\mu_B n}{2\epsilon_F}\int_{-\mu_B H}^{\mu_B H} dE\, \frac{1}{e^{\beta(E-\mu)}+1}\,\frac{e^{-\beta(E-\mu)}}{e^{-\beta(E-\mu)}}$$

$$= -\frac{\mu_B n}{2\epsilon_F} k_B T \ln\left[1 + e^{-\beta(E-\mu)}\right]\Bigg|_{-\mu_B H}^{\mu_B H}$$

$$M = \frac{\mu_B n}{2\beta\epsilon_F}\ln\left[\frac{1+e^{+\beta(\mu+\mu_B H)}}{1+e^{\beta(\mu-\mu_B H)}}\right] \quad (1)$$



$\mu - \mu_B H \quad \mu \quad \mu + \mu_B H$

⑤ Using directly (1)

$$M = \frac{\mu_B n}{2\beta\epsilon_F} \ln\left[\frac{1+e^{\beta\mu(1+\beta\mu_B H)}}{1+e^{\beta\mu(1-\beta\mu_B H)}}\right]$$

$$= \frac{\mu_B n}{2\beta\epsilon_F} \ln\left[\frac{1+e^{\beta\mu}\beta\mu_B H/(1+e^{\beta\mu})}{1-e^{\beta\mu}\beta\mu_B H/(1+e^{\beta\mu})}\right]$$

$$\simeq \frac{\mu_B n}{\cancel{2\beta}\epsilon_F} \frac{\cancel{2\beta}\mu_B H}{1+e^{\beta\mu}} e^{\beta\mu} = \frac{\mu_B^2 n}{\epsilon_F} \frac{H}{1+e^{-\beta\mu}}$$

$\beta\epsilon_F >> 1 \qquad \mu \simeq \epsilon_F$

$$M = \mu_B^2 g(\epsilon_F) \frac{1}{1+e^{-\beta\epsilon_F}} H$$

$$\boxed{M = \mu_B^2 g(\epsilon_F) H}$$

⑥

$$\chi = \mu_B^2 g(\epsilon_F) \frac{1}{1+e^{-\beta\epsilon_F}} \simeq \mu_B^2 g(\epsilon_F)$$