COMPITINO 14-11-20

ESERCIZIO 1.

① $Q_N(V,T) = \frac{q^N}{N!}$, $q = \frac{1}{h^3}\int_{R^3} d\bar{p} \int_{V} d\bar{r}\, e^{-\beta \mathcal{H}_1}$, $\mathcal{H}_1 = \frac{p^2}{2m} - \nu_0 \cos\vartheta$.

$$q = \frac{1}{\lambda^3}\int_{0\le r\le R} d\bar{r}\, e^{+\beta\nu_0\cos\vartheta} = \frac{1}{\lambda^3}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \int_0^R dr\, r^2\, e^{+\beta\nu_0\cos\vartheta} = \frac{2\pi R^3}{3\lambda^3}\int_{-1}^{1} d\cos\vartheta\, e^{+\beta\nu_0\cos\vartheta}$$

$$= \frac{4\pi R^3}{3\lambda^3}\frac{\sinh(\beta\nu_0)}{\beta\nu_0} = \frac{V}{\lambda^3}\frac{\sinh(\beta\nu_0)}{\beta\nu_0}$$

$$A = -N k_B T \ln\left(\frac{qe}{N}\right) = -N k_B T \ln\left[\frac{(V/N)e}{\lambda^3}\frac{\sinh(\beta\nu_0)}{\beta\nu_0}\right]$$

② $$E = \frac{\partial \beta A}{\partial \beta} = -\frac{\partial}{\partial\beta} N \ln\left[\frac{(V/N)e}{C\beta^{3/2}}\frac{\sinh(\beta\nu_0)}{\beta\nu_0}\right] = \frac{3N}{2\beta} - N\nu_0\coth(\beta\nu_0) + \frac{N}{\beta}$$

$$E = N k_B T\left[\frac{5}{2} - \beta\nu_0\coth(\beta\nu_0)\right]$$

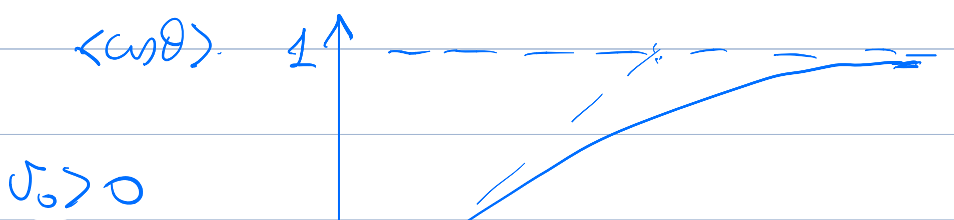
③ $$S = \frac{E-A}{T} = N k_B\left[\frac{5}{2} - \beta\nu_0\coth(\beta\nu_0) + \ln\left(\frac{(V/N)e}{\lambda^3}\frac{\sinh(\beta\nu_0)}{\beta\nu_0}\right)\right]$$

④ $$\langle\cos\vartheta\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle\cos\vartheta_i\rangle = \langle\cos\vartheta_1\rangle = -\frac{1}{\beta\nu_0} + \coth(\beta\nu_0)$$

$$\langle\cos\vartheta_1\rangle = \frac{\int_{-1}^{1} d\cos\vartheta_1 \cos\vartheta_1\, e^{+\beta\nu_0\cos\vartheta_1}}{\int_{-1}^{1} d\cos\vartheta_1\, e^{+\beta\nu_0\cos\vartheta_1}} = \frac{-2\left[\frac{\sinh(\beta\nu_0)}{(\beta\nu_0)^2} - \frac{\cosh(\beta\nu_0)}{(\beta\nu_0)}\right]}{2\frac{\sinh(\beta\nu_0)}{\beta\nu_0}} = -\frac{1}{\beta\nu_0} + \coth(\beta\nu_0)$$

$$\int_{-1}^{1} dx\, x\, e^{+\alpha x} = +\frac{\partial}{\partial\alpha}\int_{-1}^{1} dx\, e^{+\alpha x} = +\frac{\partial}{\partial\alpha}\left[2\frac{\sinh(\alpha)}{\alpha}\right] = -2\left[\frac{\sinh(\alpha)}{\alpha^2} - \frac{\cosh(\alpha)}{\alpha}\right]$$

$$\begin{cases}\coth(x) \simeq \frac{1}{x} + \frac{x}{3} \\ x \ll 1\end{cases}$$



ESERCIZIO 2

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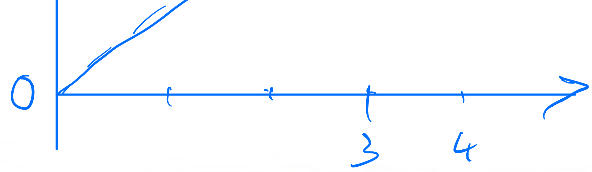


$$Q_{N_1 N_2} = \frac{1}{N_1! h^{3N_1}} q_1^{N_1} \cdot \frac{1}{N_2! h^{3N_2}} q_2^{N_2}$$

$$q_\alpha = \int d\bar{p}\, e^{-\frac{\beta p^2}{2m_\alpha}} \int_{A_b} ds \int_0^{L_z} dz\, e^{-z/l_\alpha}$$

$$l_\alpha = \frac{k_B T}{m_\alpha g} \qquad \lambda_\alpha^2 = \frac{h^2}{2\pi m_\alpha k_B T}$$

$$q_\alpha = \frac{1}{\lambda_\alpha^{3N_\alpha}} A_b l_\alpha \left(1 - e^{-L_z/l_\alpha}\right)$$

$$Q_{N_1 N_2} = \prod_{\alpha=1}^{2} \frac{1}{N_\alpha! \lambda_\alpha^{3N_\alpha}} \left[A_b l_\alpha \left(1 - e^{-L_z/l_\alpha}\right)\right]^{N_\alpha}$$

$$\simeq \prod_\alpha \left[\frac{e}{\lambda_\alpha^3 N_\alpha} A_b l_\alpha \left(1 - e^{-L_z/l_\alpha}\right)\right]^{N_\alpha}$$

$$A = -\sum_{\alpha=1}^{2} N_\alpha k_B T \ln\left[\frac{e A_b l_\alpha}{\lambda_\alpha^3 N_\alpha} \left(1 - e^{-L_z/l_\alpha}\right)\right]$$

(2)

$$P = -\frac{\partial A}{\partial V} = -\frac{1}{A_b}\frac{\partial A}{\partial L_z} =$$

$$\sum_{\alpha=1}^{2} \frac{N_\alpha k_B T}{A_b L_z} \frac{L_z}{l_\alpha} \frac{e^{-L_z/l_\alpha}}{1-e^{-L_z/l_\alpha}}$$

(3)

$$S = -\frac{\partial A}{\partial T}$$

$$= \sum_{\alpha=1}^{2} N_\alpha k_B \ln\left[\frac{e A_b l_\alpha}{\lambda_\alpha^3 N_\alpha}\left(1-e^{-L_z/l_\alpha}\right)\right]$$

$$+ \sum_{\alpha=1}^{2} N_\alpha k_B T \frac{\partial}{\partial T} \ln\left[T^{5/2}\left(1-e^{-L_z m_\alpha g/(k_B T)}\right)\right]$$

$$= \sum_{\alpha=1}^{2} \Big\{ N_\alpha k_B \ln\left[\frac{e A_b l_\alpha}{\lambda_\alpha^3 N_\alpha}\left(1-e^{-L_z/l_\alpha}\right)\right]$$

$$+ \frac{5}{2} N_\alpha k_B - N_\alpha k_B \frac{L_z}{l_\alpha} \frac{e^{-L_z/l_\alpha}}{1-e^{-L_z/l_\alpha}} \Big\}$$

$$S = k_B \sum_{\alpha=1}^{2} N_\alpha \Big\{ \ln\left[\frac{e A_b l_\alpha}{\lambda_\alpha^3 N_\alpha}\left(1-e^{-L_z/l_\alpha}\right)\right]$$

$$+ \frac{5}{2} - \frac{L_z}{l_\alpha} \frac{e^{-L_z/l_\alpha}}{1-e^{-L_z/l_\alpha}} \Big\}$$

④ $$\rho_\alpha(z) = N_\alpha \frac{\int dr_1^{(\alpha)} e^{-\beta m_\alpha g z_1^{(\alpha)}} \delta(r-r_1^{(\alpha)})}{\int dr_1^{(\alpha)} e^{-\beta m g z_1^{(\alpha)}}}$$

$$= N_\alpha \frac{e^{-\beta m_\alpha g z}}{A_b \ell_\alpha (1 - e^{-\beta m_\alpha g L_z}}$$

$$\rho_\alpha(z) = N_\alpha \frac{e^{-z/\ell_\alpha}}{A_b \ell_\alpha (1 - e^{-L_z/\ell_\alpha})}$$

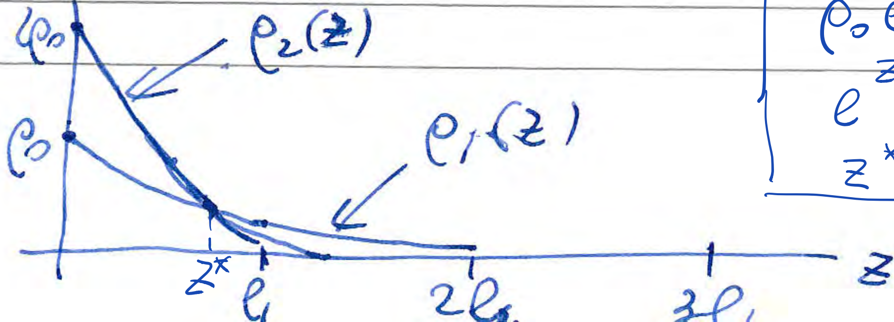
$$\rho_\alpha(z) \simeq \frac{N_\alpha}{V} \frac{L_z}{\ell_\alpha} e^{-z/\ell_\alpha}$$ $L_z \gg \ell_\alpha \Rightarrow e^{-L_z/\ell_\alpha} \ll 1$

$$\rho_2(z) = \frac{N_2}{V} \frac{L_z}{\ell_2} e^{-z/\ell_2}$$ $m_2 = 2m_1 \Rightarrow \ell_2 = \frac{\ell_1}{2}$

$$= \frac{N_1}{V} \frac{2L_z}{\ell_1} e^{-\frac{2z}{\ell_1}}$$

$$\rho_1 = \frac{N_1}{V} \frac{L_z}{\ell_1} e^{-z/\ell_1} = \rho_0 e^{-z/\ell_1}$$

$$\rho_2 = 2\rho_0 e^{-2z/\ell_1}$$



$\rho_1(z^*) = \rho_2(z^*) \Rightarrow$

$\rho_0 e^{-z^*/\ell_1} = 2\rho_0 e^{-2z^*/\ell_1}$

$e^{z^*/\ell_1} = 2$

$z^* = \ell_1 \ln 2$