ESERCIZIO 1

①
$$\mathcal{H} = \sum_{i=1}^{N} \mathcal{H}_1(\bar{p}_i, \bar{r}_i) = \sum_{i=1}^{N} (c|\bar{p}_i| + mgz_i)$$

$$Q_N(V,T;c,m) = \frac{1}{N!\,h^{3N}} \int d^{3N}q\, d^{3N}p\, e^{-\sum_i \beta(c|\bar{p}_i| + mgz_i)}$$

$$-k_B T\, m \frac{\partial \ln Q_N}{\partial m} =$$

$$= \frac{\frac{1}{N!\,h^{3N}} \int d^{3N}q \int d^{3N}p\, e^{-\sum_i \beta(c|\bar{p}_i| + mgz_i)} (-\cancel{k_B T}\, m) \sum_{i=1}^{N} (-\cancel{\beta} g z_i)}{\frac{1}{N!\,h^{3N}} \int d^{3N}q \int d^{3N}p\, e^{-\sum_i \beta(c|\bar{p}_i| + mgz_i)}}$$

$$= \left\langle \sum_i mgz_i \right\rangle = U$$

②
$$-k_B T c \frac{\partial \ln Q_N}{\partial c}$$

$$= \frac{\frac{1}{N!\,h^{3N}} \int d^{3N}q \int d^{3N}p\, e^{-\sum_{i=1}^{N} \beta(c|\bar{p}_i| + mgz_i)} \left(-\cancel{k_B T}\, c \sum_{i=1}^{N} (-\cancel{\beta} |\bar{p}_i|)\right)}{\frac{1}{N!\,h^{3N}} \int d^{3N}q \int d^{3N}p\, e^{-\sum_i \beta(c|\bar{p}_i| + mgz_i)}}$$

$$= \left\langle \sum_i c|\bar{p}_i| \right\rangle = K$$

③ $$Q_N(V, T; c, m) = \frac{q}{N!} \qquad \ell = \frac{k_B T}{mg}$$

$$q = \int \frac{d\bar{p}}{h^3} \int d\bar{r}\; e^{-\beta c p}\, e^{-\beta m g z}$$

$$= \frac{4\pi}{h^3} \int_0^\infty dp\, p^2 e^{-\beta c p} \int_{A_b} dS \int_0^{L_z} e^{-\beta m g z}\, dz$$

$$= \frac{4\pi}{(h\beta c)^3} \int_0^\infty dt\, t^2 e^{-t}\; A_b \int_0^{L_z} dz\, e^{-z/\ell}$$

$$= \frac{4\pi}{(h\beta c)}\, \Gamma(3)\, A_b \left(-\ell e^{-z/\ell}\right)\Big|_0^{L_z}$$

$$q = \frac{8\pi}{(h\beta c)^3} A_b \ell \left(1 - e^{-L_z/\ell}\right)$$

$$\boxed{Q_N = \frac{1}{N!}\left[\frac{8\pi A_b \ell}{(h\beta c)^3}\left(1 - e^{-L_z/\ell}\right)\right]^N}$$

$$U = -k_B T m \frac{\partial}{\partial m} \ln Q_N$$

$$= -k_B T m \frac{\partial}{\partial m} N\left[\ln \frac{k_B T}{mg} + \ln\left(1 - e^{-\frac{mgL_z}{k_B T}}\right)\right]$$

$$= -k_B T \left[-m N \frac{1}{m} + \frac{e^{-L_z/\ell}}{1 - e^{-L_z/\ell}} \frac{L_z}{\ell} \right]$$

$$\boxed{U = N k_B T \left[1 - \frac{L_z/\ell}{e^{L_z/\ell} - 1} \right]}$$

$$K = -k_B T c \frac{\partial \ln Q_N}{\partial c} = -k_B T c \frac{\partial}{\partial c} N \ln \frac{1}{c^3}$$

$$= -k_B T N c \left(-\frac{3}{c} \right) = 3 N k_B T$$

$$\boxed{K = 3 N k_B T}$$

4) $U = N k_B T \left[1 - \frac{y}{e^y - 1} \right]$; $y = L_z/\ell$

i) $L_z/\ell >> 1 \Rightarrow y >> 1$

$$U \simeq N k_B T \left[1 - \frac{y}{e^y} \right] = N k_B T \left[1 - y e^{-y} \right]$$

$$\boxed{U \simeq N k_B T}$$

c) $L_z/\ell << 1 \Rightarrow y < 1$

$$1 - \frac{y}{e^y - 1} \simeq 1 - \frac{y}{y + \frac{y^2}{2} + o(y^3)} = 1 - \frac{1}{1 + \frac{y}{2} + o(y^2)}$$

$$= \frac{1 + y/2 - 1 + o(y^2)}{1 + \frac{y}{2} + o(y^2)} = \frac{y/2 + o(y^2)}{1 + \frac{y}{2}}$$

$$= \frac{y}{2} + o(y^2)$$

$$U \simeq N k_B T \frac{y}{2} = N \cancel{k_B T} \frac{L_z m g}{\cancel{k_B T}} \frac{1}{2} = \frac{N m g L_z}{2}$$

$$\boxed{U \simeq \frac{N m g L_z}{2}}$$

ESERCIZIO 2

① $$\tilde{Z}(N,P,T) = \beta P \int_0^\infty dV\, e^{-\beta P V} Q_N(V,T)$$

$$= \beta P \int_0^\infty dV\, e^{-\beta P V} \frac{V^N}{\lambda_T^{3N} N!} = \frac{1}{\lambda_T^{3N} N!} \frac{\beta P}{(\beta P)^{N+1}} \int_0^\infty dt\, e^{-t} t^N$$

$$\boxed{t = \beta P V}$$

$$\tilde{Z}(N,P,T) = \frac{\Gamma(N+1)}{\lambda_T^{3N} N! (\beta P)^N}$$

$$\tilde{Z}(N,P,T) = \frac{1}{\lambda_T^{3N} (\beta P)^N}$$

$$G = -k_B T \ln \frac{1}{(\lambda_T^3 \beta P)^N} = N k_B T \ln(\lambda_T^3 \beta P)$$

$$\boxed{G = N k_B T \ln(\lambda_T^3 \beta P)}$$

② $$dG = -S\,dT + V\,dP + \mu\,dN$$

$$\mu = \left.\frac{\partial G}{\partial N}\right|_{T,P} = k_B T \ln(\lambda_T^3 \beta P)$$

③ $V = \left.\frac{\partial G}{\partial \underline{P}}\right|_{N,T} = N k_B T \cdot \frac{1}{\underline{P}}$

$\Rightarrow \underline{P} = \frac{N k_B T}{V} = \rho k_B T$

④ $\mu = k_B T \ln(\lambda_T^3 \beta \underline{P}) = k_B T \ln(\lambda_T^3 \rho)$