I - Compito 17.11.17 ESERCIZIO 1

① 6 punti

$$\mathcal{H} = \sum_{i=1}^{N} \left[\frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 \right], \quad \text{separabile}$$

$$Q_N(T) = \frac{1}{N!} q^N, \quad q = \frac{1}{h} \int_{-\infty}^{\infty} dp\, e^{-\frac{\beta p^2}{2m}} \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta m \omega^2}{2} x^2} \equiv q_1 \cdot q_2$$

$$q_1 = \frac{1}{h} \int_{-\infty}^{\infty} dp\, e^{-\frac{\beta p^2}{2m}} = \frac{\sqrt{2mK_BT}}{h} \int_{-\infty}^{\infty} dy\, e^{-y^2} = \frac{\sqrt{2mK_BT}}{h} \sqrt{\pi} = \frac{1}{\lambda}$$

$$q_2 = \int_{-\infty}^{\infty} dx\, e^{-\frac{\beta m \omega^2}{2} x^2} = \sqrt{\frac{2K_BT}{m\omega^2}} \int_{-\infty}^{\infty} ds\, e^{-s^2} = \sqrt{\frac{2K_BT}{m}} \frac{\sqrt{\pi}}{\omega} = \sqrt{\frac{2\pi K_BT}{m}} \frac{1}{\omega}$$

$$y = \sqrt{\frac{1}{2mK_BT}}\, p \Rightarrow dp = \sqrt{2mK_BT}\, dy$$

$$s = \sqrt{\frac{m}{2K_BT}}\, \omega\, dx \Rightarrow dx = \sqrt{\frac{2K_BT}{m}} \frac{ds}{\omega}$$

$$q = \frac{\sqrt{2\pi m K_BT}}{h} \cdot \sqrt{\frac{2\pi K_BT}{m}} \frac{1}{\omega} = \frac{2\pi K_BT}{h\omega} = \frac{K_BT}{\hbar\omega}$$

$$Q_N(T) = \frac{1}{N!} \left(\frac{K_BT}{\hbar\omega} \right)^N$$

$$A(N,T) = -K_BT \ln\left[\frac{1}{N!} \left(\frac{K_BT}{\hbar\omega} \right)^N \right] \simeq -K_BT \ln\left[\left(\frac{e}{N} \right)^N \left(\frac{K_BT}{\hbar\omega} \right)^N \right]$$

$$\boxed{A(N,T) \simeq -NK_BT \ln\left[\frac{e}{N} \frac{K_BT}{\hbar\omega} \right]}$$

② 3 p.

$$E = \frac{\partial \beta A(N,\beta)}{\partial \beta}\Big|_N = -\frac{\partial}{\partial\beta} N \ln\left[\frac{e}{N}\frac{1}{\beta\hbar\omega}\right] = -\frac{\partial}{\partial\beta} N \ln\frac{1}{\beta}$$

$$= N\frac{\partial}{\partial\beta}\ln\beta = \frac{N}{\beta} = Nk_BT$$

$$\boxed{E = Nk_BT}$$

③ 3 p.

$$S(N,T) = \frac{1}{T}\left[E - A\right] = \frac{1}{T}\left[Nk_BT + Nk_BT\ln\left[\frac{e}{N}\frac{k_BT}{\hbar\omega}\right]\right]$$

$$= Nk_B\left[1 + \ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right)\right] = Nk_B\left[\ln(e) + \ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right)\right]$$

$$\boxed{S(N,T) = Nk_B\ln\left(\frac{e^2}{N}\frac{k_BT}{\hbar\omega}\right)}$$

④ 3 p.

$$\mu = \frac{\partial}{\partial N}A(N,T)\Big|_T = \frac{\partial}{\partial N}\left[-Nk_BT\ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right)\right]$$

$$= -k_BT\ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right) - Nk_BT\frac{\partial}{\partial N}\ln\left(\frac{1}{N}\right) = -k_BT\left[\ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right) - N\cdot\frac{1}{N}\right]$$

$$= -k_BT\left[\ln\left(\frac{e}{N}\frac{k_BT}{\hbar\omega}\right) - \ln(e)\right]$$

$$\boxed{\mu = k_BT\ln\left(\frac{N\hbar\omega}{k_BT}\right)}$$

ESERCIZIO 2

① 6p.

$$\Sigma(E,N) = \frac{1}{N!\,h^N}\int dp_1 \dots dp_N \int dq_1 \dots dq_N \,, \quad \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + \frac{m\omega^2}{2} q_i^2\right] \leq E$$

Con il cambio di variabili suggerito si ottiene

$$\Sigma(E,N) = \frac{1}{N!\,h^N}(2m)^{N/2}\left(\frac{2}{m}\right)^{N/2}\frac{1}{\omega^N}\int dx_1 \dots dx_{2N} \,, \quad \sum_{i=1}^{2N} x_i^2 \leq E$$

$\Omega_{2N} = \int dx_1 \dots dx_{2N}$, $\sum_{i=1}^{2N} x_i^2 \leq E$ è il volume di una ipersfera

in $2N$ dimensioni di raggio $R = \sqrt{E}$, $\Omega_{2N} = \frac{\pi^N}{\Gamma(N+1)}(\sqrt{E})^{2N}$

$$\Omega_{2N} = \frac{(\pi E)^N}{N!}, \quad \Sigma(E,N) = \frac{1}{N!}\left(\frac{2}{h\omega}\right)^N \cdot \frac{(\pi E)^N}{N!} = \left(\frac{1}{N!}\right)^2\left(\frac{2\pi E}{h\omega}\right)$$

$$S(E,N) = k_B \ln\left[\left(\frac{1}{N!}\right)^2\left(\frac{2\pi E}{h\omega}\right)^N\right] \simeq k_B \ln\left[\left(\frac{e}{N}\right)^{2N}\left(\frac{2\pi E}{h\omega}\right)^N\right]$$

$$\boxed{S(E,N) = N k_B \ln\left[\frac{2\pi e^2 E}{N^2 h\omega}\right]}$$

② 3p.

$$\frac{1}{T} = \left.\frac{\partial S(E,N)}{\partial E}\right|_N = N k_B \frac{\partial}{\partial E}\ln(E) = \frac{N k_B}{E}$$

$$\boxed{E = N k_B T}$$

③ 3p.

$$S(N,T) = N k_B \ln\left[\frac{2\pi e^2 E}{N^2 h\omega}\right] = N k_B \ln\left[\frac{e^2 k_B T}{N \hbar\omega}\right]$$

$$A(N,T) = E(N,T) - T S(N,T) = N k_B T\left[1 - \ln\left(\frac{e^2 k_B T}{N\hbar\omega}\right)\right]$$

$$\boxed{A(N,T) = -N k_B T \ln\left[\frac{e\, k_B T}{N\hbar\omega}\right]}$$

④ 3p.

Identica alla ④ dell'ESERCIZIO 1.