SCRITTO DEL 13.11.2015 $\boxed{\text{FS}}$

Esercizio 1

① $$Q_N(V,T) = \frac{1}{N!\,h^{3N}}\left[\int d\bar{p}\int d\bar{r}\; e^{-\beta h(p,q)}\right]^N$$

$$= \frac{1}{N!}\, q^N$$

$$q = \int dV \int \frac{d\bar{p}}{h^3}\, e^{-\beta c p} = \frac{V}{h^3}\, 4\pi \int_0^\infty dp\, p^2 e^{-\beta c p}$$

$$\int_0^\infty dx\, x^2 e^{-\alpha x} = \frac{\partial^2}{\partial \alpha^2}\int_0^\infty dx\, e^{-\alpha x} = \frac{\partial^2}{\partial \alpha^2}\left[-\frac{e^{-\alpha x}}{\alpha}\right]_0^\infty$$
$$= \frac{\partial^2}{\partial \alpha^2}\frac{1}{\alpha} = \frac{2}{\alpha^3}$$

$$q = \frac{8\pi}{(h\beta c)^3}\, V$$

$$\boxed{Q = \frac{1}{N!}\left[\frac{8\pi V}{(h\beta c)^3}\right]^N \simeq \left(\frac{V}{N}\, \frac{e\, 8\pi}{h\beta c}\right)^N}$$

② $$\mathcal{Z}(\mu,V,T) = \sum_{N=0}^{\infty} e^{\beta\mu N} Q_N = \sum_{N=0}^{\infty} \frac{(e^{\beta\mu} q)^N}{N!}$$

$$\mathcal{Z}(\mu,V,T) = \exp\left[e^{\beta\mu}\, \frac{8\pi V}{(h\beta c)^3}\right]$$

(3)

$$\ln Z = \beta P V = e^{\beta \mu} \frac{8\pi V}{(h\beta c)^3}$$

$$e^{\beta\mu} = \frac{(h\beta c)^3}{8\pi} \beta P$$

$$\mu = k_B T \ln \frac{(h\beta c)^3 \beta P}{8\pi}$$

(4)

$$\mu = \frac{\partial A}{\partial N} = \frac{\partial}{\partial N}\left[- k_B T N \ln \left(\frac{V e 8\pi}{N (h\beta c)^3} \right) \right]$$

$$= - k_B T \ln \frac{V e 8\pi}{N (h\beta c)^3} + k_B T$$

$$\beta\mu = \ln \frac{N (h\beta c)^3}{V 8\pi} = \ln \frac{(h\beta c)^3 \beta P}{8\pi}$$

$$\Rightarrow \beta P = \frac{N}{V} = \rho$$

ESERCIZIO 2

① $h = p^2/2m - \bar{d}\cdot\bar{E}$

$Q_N(V,T) = \frac{1}{N!} q^N$, $q = \frac{1}{h^3}\int_V d\bar{r}\int d\bar{p}\, e^{-\beta p^2/2m}\int_{-1}^{1} d\cos\gamma\, e^{\beta d E\cos\gamma}$

$q = \frac{V}{\lambda^3}\int_{-1}^{1} dy\, e^{\beta d E y} = \frac{V}{\lambda^3}\frac{1}{\beta d E}2\sinh(\beta d E)$

② $Q_N(V,T) = \frac{1}{N!}\left(\frac{V}{\lambda^3}\frac{2}{\beta d E}\sinh(\beta d E)\right)^N \simeq \left(\frac{eV}{N}\frac{2}{\lambda^3}\sinh(\beta d E)\right)^N$

$A(N,V,T) = -N k_B T \ln\left[\frac{eV}{N}\frac{2}{\lambda^3}\sinh(\beta d E)\right]$, $\lambda^2 = h^2/(2\pi m k_B T)$

③ $P = \langle \sum_i \bar{d}_i\cdot\hat{z}\rangle/V = N\langle\bar{d}\cdot\hat{z}\rangle/V = \frac{N}{V}\langle d\cos\gamma\rangle = \frac{N}{V} d\langle\cos\gamma\rangle$

$\langle\cos\gamma\rangle = \int_{-1}^{1} d\cos\gamma\, e^{\beta d E\cos\gamma}\cos\gamma \Big/ \int_{-1}^{1} d\cos\gamma\, e^{\beta d E\cos\gamma} = \int_{-1}^{1} dy\, y\, e^{\beta d E y}\Big/\int_{-1}^{1} dy\, e^{\beta d E y} =$

$\frac{\partial}{\partial\alpha}\ln\left[\int_{-1}^{1} dy\, e^{\alpha y}\right]_{\alpha=\beta d E} = \frac{\partial}{\partial\alpha}\ln\left[\frac{2}{\alpha}\sinh(\alpha)\right]\Big|_{\alpha=\beta d E} =$

$\frac{\partial}{\partial\alpha}\left[\ln(\sinh\alpha) - \ln\alpha\right]_{\alpha=\beta d E} = \left[\coth(\alpha) - \frac{1}{\alpha}\right]_{\alpha=\beta d E}$

$P = \frac{N}{V} d\left[\coth(\beta d E) - \frac{1}{\beta d E}\right]$

④ $E \to$ $\beta d E \ll 1$, $\coth(\beta d E) \simeq \frac{1}{\beta d E} + \frac{\beta d E}{3}$

$P \simeq \frac{N}{V} d\frac{\beta d E}{3} = \alpha = \frac{N}{V}\frac{d^2}{3 k_B T}$

$$\coth(\alpha) = \frac{\cosh(\alpha)}{\sinh(\alpha)} \simeq \frac{1 + \frac{\alpha^2}{2}}{\alpha + \frac{\alpha^3}{6}} = \frac{1}{\alpha} \frac{1 + \frac{\alpha^2}{2}}{1 + \frac{\alpha^2}{6}} \simeq \frac{1}{\alpha}\left(1 + \frac{\alpha^2}{2}\right)\left(1 - \frac{\alpha^2}{6}\right)$$

$$\simeq \frac{1}{\alpha}\left(1 + \frac{\alpha^2}{2} - \frac{\alpha^2}{6}\right) = \frac{1}{\alpha}\left(1 + \frac{\alpha}{3}\right)$$