(b) $z \gg 1$, ovvero $\beta\mu \gg 1$ $(z = e^{\beta\mu})$.

Vogliamo calcolare

$$\rho = \frac{1}{V}\sum_{\vec{p}} \frac{1}{e^{\beta(\epsilon_{\vec{p}}-\mu)}+1} \equiv \int d\epsilon \; g(\epsilon)\, n(\epsilon) \qquad (138)$$

$$g(\epsilon) = \frac{1}{V}\sum_{\vec{p}} \delta(\epsilon - \epsilon_{\vec{p}}) \;, \quad n(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \,. \qquad (139)$$

Consideriamo più in generale

$$I = \int_0^\infty d\epsilon \; f(\epsilon)\, n(\epsilon) \qquad (140)$$

con $f(\epsilon)$ una funzione "smooth" e lenta rispetto all'esponenziale. Nota che $\epsilon_{\vec{p}} = p^2/2m \geq 0$ implica che $g(\epsilon) \neq 0$ solo per $\epsilon \geq 0$

Allora

$$\beta I = \beta \int_0^\infty d\epsilon \; f(\epsilon) \frac{1}{e^{\beta(\epsilon-\mu)}+1} \qquad (141) \qquad \boxed{\begin{array}{c} t = \beta(\epsilon-\mu) \\ \hline \epsilon = \mu + k_B T t \end{array}}$$

$$= \int_{-\beta\mu}^{\infty} dt \; f(\mu + k_B T t) \frac{1}{e^t+1}$$

$$= \int_0^\infty dt \, \frac{f(\mu + k_B T t)}{e^t+1} + \int_{-\beta\mu}^{0} dt \, \frac{f(\mu + k_B T t)}{e^t+1}$$

$$= \int_0^\infty dt \, \frac{f(\mu + k_B T t)}{e^t+1} + \int_0^{\beta\mu} ds \, \frac{f(\mu - k_B T s)}{e^{-s}+1} \qquad (142) \qquad \boxed{s = -t}$$

Notiamo che essendo $1/(1+e^{-s}) = 1 - 1/(1+e^{s})$

$$\beta I_2 = \int_0^{\beta\mu} ds \; \frac{f(\mu - k_B T s)}{e^{-s}+1} = \int_0^{\beta\mu} ds\, f(\mu - k_B T s) - \int_0^{\beta\mu} ds\, \frac{f(\mu - k_B T s)}{e^{s}+1} \tag{143}$$

Ponendo $\epsilon = \mu - k_B T s$ e $s = t$, otteniamo

$$\beta I_2 = -\beta \int_\mu^0 d\epsilon\, f(\epsilon) - \int_0^{\beta\mu} dt \; \frac{f(\mu - k_B T t)}{e^{t}+1}$$

$$= \beta \int_0^\mu d\epsilon\, f(\epsilon) - \int_0^\infty dt\; \frac{f(\mu - k_B T t)}{e^{t}+1} + O(e^{-\beta\mu}). \tag{144}$$

$$\simeq \beta \int_0^\mu d\epsilon\, f(\epsilon) - \int_0^\infty dt\; \frac{f(\mu - k_B T t)}{e^{t}+1} \tag{145}$$

Per ottenere la (145) abbiamo usato

$$\left| \int_{\beta\mu}^\infty dt\; \frac{f(\mu - k_B T t)}{e^{t}+1} \right| < \int_{\beta\mu}^\infty dt\; \frac{|f(\mu - k_B T t)|}{e^{t}}$$

$$= -\beta \int_0^{-\infty} d\epsilon\, |f(\epsilon)|\, e^{-\beta(\mu-\epsilon)} = \beta \int_0^\infty d\epsilon\, |f(\epsilon)|\, e^{-\beta\epsilon} \cdot e^{-\beta\mu} \tag{146}$$

Combinando la (145) con la (142) otteniamo

$$\beta I = \beta\int_0^\mu d\epsilon\, f(\epsilon) + \int_0^\infty dt\, \frac{1}{e^t+1}\left(f(\mu+k_BTt)-f(\mu-k_BTt)\right)$$

$$= \beta\int_0^\mu d\epsilon\, f(\epsilon) + \int_0^\infty dt\, \frac{1}{e^t+1}\left[2f'(\mu)(k_BT)t + \frac{2}{3!}f'''(\mu)(k_BT)^3t^3 + \dots\right]. \quad (147)$$

Quindi possiamo scrivere, notando che

$$2\int_0^\infty dt\, \frac{t}{e^t+1} = \frac{\pi^2}{6}\,; \quad \frac{1}{3}\int_0^\infty dt\, \frac{t^3}{e^t+1} = \frac{7\pi^4}{360}, \quad (148)$$

$$I = \int_0^\infty d\epsilon\, \frac{f(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} = \int_0^\mu d\epsilon\, f(\epsilon) + \frac{\pi^2}{6}(k_BT)^2 f'(\mu) + \frac{7\pi^4}{360}(k_BT)^4 f'''(\mu) + \dots \quad (149)$$

$\beta\mu >> 1$

Applicando la (149) alla (138) otteniamo

$$\rho = \int_0^\mu d\epsilon\, g(\epsilon) + \frac{\pi^2}{6}(k_BT)^2 g'(\mu) + \dots \quad (150)$$

Inoltre

$$\int_0^\infty d\epsilon\, \frac{g(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} = \frac{1}{\lambda^3}\frac{4}{\sqrt{\pi}}\int_0^\infty dx\, \frac{x^2}{\frac{e^{x^2}}{z}+1} \quad (151)$$

Mostro che possiamo scrivere

$$g(\epsilon) = \frac{\beta^{3/2}}{\lambda^3}\frac{2}{\sqrt{\pi}}\sqrt{\epsilon} \qquad (152)$$

E quindi

$$\rho = \frac{\beta^{3/2}}{\lambda^3}\frac{4}{3\sqrt{\pi}}\mu^{3/2} + \frac{\pi^2}{6}\frac{1}{\beta^2}\frac{\beta^{3/2}}{\lambda^3}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\mu}} + \dots$$

$$\lambda^3\rho = \frac{4}{3\sqrt{\pi}}(\beta\mu)^{3/2} + \frac{\pi^2}{6\sqrt{\pi}}\frac{1}{\sqrt{\beta\mu}} + \dots$$

$$\lambda^3\rho = \frac{4}{3\sqrt{\pi}}(\beta\mu)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{(\beta\mu)^2} + \dots\right] \qquad (153)$$

Ad ordine dominante in $\beta\mu = \ln z$ otteniamo

$$\frac{\mu}{k_B T} = \left(\frac{3\sqrt{\pi}}{4}\rho\right)^{2/3}\lambda^2 = \left(\frac{3\sqrt{\pi}}{4}\rho\right)^{2/3}\frac{h^2}{2\pi m k_B T}$$

Quindi

$$\mu = 4\pi\frac{\hbar^2}{2m}\left(\frac{3\sqrt{\pi}}{4}\rho\right)^{2/3} = \frac{\hbar^2}{2m}(6\pi^2\rho)^{2/3}$$

Ovvero

$$\begin{cases} \mu = \frac{\hbar^2}{2m}(6\pi^2\rho)^{2/3} \equiv \epsilon_F \\ \lambda^3\rho \gg 1 \end{cases} \qquad (154)$$

in quanto $\beta\mu \gg 1$ implica $\lambda^3\rho \gg 1$.

Osserviamo che

$$\lambda^3\rho \sim \frac{\rho}{T^{3/2}} \gg 1$$

implica ρ grande o T piccolo e può certamente essere ottenuto quando $\rho \to \infty$ oppure $T \to 0$.

ϵ_F è indipendente da T: è chiamata energia di Fermi.

Dalla (153) possiamo calcolare la prima correzione in temperatura ad ϵ_F. Abbiamo

$$\frac{3\sqrt{\pi}}{4}\lambda^3\rho = (\beta\epsilon_F)^{3/2} = (\beta\mu)^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{(\beta\mu)^2} + \dots\right]$$

ci fornisce

$$\beta\mu = \frac{\beta\epsilon_F}{\left[1 + \frac{\pi^2}{8}\frac{1}{(\beta\mu)^2} + \dots\right]^{2/3}} \simeq \beta\epsilon_F\left[1 - \frac{2}{3}\frac{\pi^2}{8}\frac{1}{(\beta\mu)^2}\right]$$

$$\simeq \beta\epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right]$$

$$\begin{cases} \mu = \epsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2 + \dots\right] \\ T \to 0 \end{cases} \qquad (155)$$

Se definiamo T_F (la temperatura di FERMI) da

$$k_B T_F = \epsilon_F ,$$

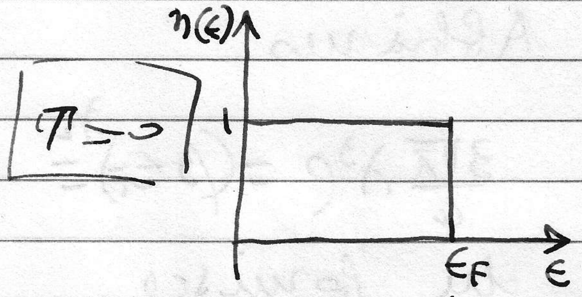
evidentemente basse temperature significa $T \ll T_F$

Nel limite $T \to 0$ abbiamo

$$\langle n_{\vec{p}} \rangle = n(\epsilon_{\vec{p}}) = \frac{1}{e^{\beta(\epsilon_{\vec{p}}-\mu)}+1} = \frac{1}{e^{\beta(\epsilon_{\vec{p}}-\epsilon_F)}+1} .$$

Chiaramente a $T=0$

$$\begin{aligned} n(\epsilon) &= 1 , \quad \epsilon < \epsilon_F \\ \text{"} &= 0 , \quad \epsilon > \epsilon_F , \end{aligned}$$



cioè ϵ_F fornisce il livello energetico occupato più alto!