$$\phi = \int_{-\infty}^{+\infty} e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\mathbf{k}} \frac{d^3k}{(2\pi)^3} \tag{51.2}$$

where d^3k denotes $dk_x dk_y dk_z$. In this formula $\phi_{\mathbf{k}} = \int \phi(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} dV$. Applying the Laplace operator to both sides of (51.2), we obtain

$$\Delta\phi = -\int_{-\infty}^{+\infty} k^2 e^{i\mathbf{k}\cdot\mathbf{r}} \phi_{\mathbf{k}} \frac{d^3k}{(2\pi)^3},$$

so that the Fourier component of the expression $\Delta\phi$ is

$$(\Delta\phi)_{\mathbf{k}} = -k^2 \phi_{\mathbf{k}}.$$

On the other hand, we can find $(\Delta\phi)_{\mathbf{k}}$ by taking Fourier components of both sides of equation (51.1),

$$(\Delta\phi)_{\mathbf{k}} = -\int 4\pi e \delta(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} dV = -4\pi e.$$

Equating the two expressions obtained for $(\Delta\phi)_{\mathbf{k}}$, we find

$$\phi_{\mathbf{k}} = \frac{4\pi e}{k^2}. \tag{51.3}$$

This formula solves our problem.

Just as for the potential ϕ, we can expand the field

$$\mathbf{E} = \int_{-\infty}^{+\infty} \mathbf{E}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3k}{(2\pi)^3}. \tag{51.4}$$

With the aid of (51.2), we have

$$\mathbf{E} = -\operatorname{grad} \int_{-\infty}^{+\infty} \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3k}{(2\pi)^3} = -\int i\mathbf{k}\phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{d^3k}{(2\pi)^3}.$$

Comparing with (51.4), we obtain

$$\mathbf{E}_{\mathbf{k}} = -i\mathbf{k}\phi_{\mathbf{k}} = -i\frac{4\pi e\mathbf{k}}{k^2}. \tag{51.5}$$

From this we see that the field of the waves, into which we have resolved the Coulomb field, is directed along the wave vector. Therefore these waves can be said to be *longitudinal*.

§ 52. Characteristic vibrations of the field

We consider an electromagnetic field (in the absence of charges) in some finite volume of space. To simplify further calculations we assume that this volume has the form of a rectangular parallelepiped with sides A, B, C, respectively. Then we can expand all quantities characterizing the field in this parallelepiped in a triple Fourier series (for the three coordinates). This expansion can be written (e.g. for the vector potential) in the form:

$$\mathbf{A} = \sum_{\mathbf{k}} \mathbf{A_k} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{52.1}$$

explicitly indicating that **A** is real. The summation extends here over all possible values of the vector **k** whose components run through the values

$$k_x = \frac{2\pi n_x}{A}, \quad k_y = \frac{2\pi n_y}{B}, \quad k_z = \frac{2\pi n_z}{C}, \tag{52.2}$$

where n_x, n_y, n_z are positive or negative integers. Since **A** is real, the coefficients in the expansion (52.1) are related by the equations $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}^*_{\mathbf{k}}$. From the equation div $\mathbf{A} = 0$ it follows that for each **k**,

$$\mathbf{k} \cdot \mathbf{A_k} = 0, \tag{52.3}$$

i.e., the complex vectors $\mathbf{A_k}$ are "perpendicular" to the corresponding wave vectors **k**. The vectors $\mathbf{A_k}$ are, of course, functions of the time; from the wave equation (46.7), they satisfy the equation

$$\ddot{\mathbf{A}}_{\mathbf{k}} + c^2 k^2 \mathbf{A_k} = 0. \tag{52.4}$$

If the dimensions A, B, C of the volume are sufficiently large, then neighbouring values of k_x, k_y, k_z (for which n_x, n_y, n_z differ by unity) are very close to one another. In this case we may speak of the number of possible values of k_x, k_y, k_z in the small intervals Δk_x, Δk_y, Δk_z.

Since to neighbouring values of, say, k_x, there correspond values of n_x differing by unity, the number Δn_x of possible values of k_x in the interval Δk_x is equal simply to the number of values of n_x in the corresponding interval. Thus, we obtain

$$\Delta n_x = \frac{A}{2\pi}\Delta k_x, \quad \Delta n_y = \frac{B}{2\pi}\Delta k_y, \quad \Delta n_z = \frac{C}{2\pi}\Delta k_z.$$

The total number Δn of possible values of the vector **k** with components in the intervals Δk_x, Δk_y, Δk_z is equal to the product $\Delta n_x \, \Delta n_y \, \Delta n_z$, that is,

$$\Delta n = \frac{V}{(2\pi)^3}\Delta k_x \Delta k_y \Delta k_z, \tag{52.5}$$

where $V = ABC$ is the volume of the field. It is easy to determine from this the number of possible values of the wave vector having absolute values in the interval Δk, and directed into the element of solid angle Δo. To get this we need only transform to polar coordinates in the "k space" and write in place of $\Delta k_x \Delta k_y \Delta k_z$ the element of volume in these coordinates. Thus

$$\Delta n = \frac{V}{(2\pi)^3} k^2 \Delta k \Delta o. \tag{52.6}$$

Replacing Δo by 4π, we find the number of possible values of **k** with absolute value in the interval Δk and pointing in all directions: $\Delta n = (V/2\pi^2)k^2\Delta k$.

We calculate the total energy

$$\mathscr{E} = \frac{1}{8\pi}\int (\mathbf{E}^2 + \mathbf{H}^2)\,dV$$

of the field, expressing it in terms of the quantities $\mathbf{A_k}$. For the electric and magnetic fields we have

$$\mathbf{E} = -\frac{1}{c}\dot{\mathbf{A}} = -\frac{1}{c}\sum_{\mathbf{k}} \dot{\mathbf{A}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}},$$

$$\mathbf{H} = \operatorname{curl} \mathbf{A} = i \sum_{\mathbf{k}} (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{52.7}$$

When calculating the squares of these sums, we must keep in mind that all products of terms with wave vectors $\mathbf{k}$ and $\mathbf{k}'$ such that $\mathbf{k} \neq \mathbf{k}'$ give zero on integration over the whole volume. In fact, such terms contain factors of the form $e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}}$, and the integral, e.g. of

$$\int_0^A e^{i\frac{2\pi}{A}n_x x}\, dx,$$

with integer n_x different from zero, gives zero. In those terms with $\mathbf{k}' = -\mathbf{k}$, the exponentials drop out and integration over dV gives just the volume V.

As a result, we obtain

$$\mathscr{E} = \frac{V}{8\pi} \sum_{\mathbf{k}} \left\{ \frac{1}{c^2} \dot{\mathbf{A}}_{\mathbf{k}} \cdot \dot{\mathbf{A}}_{\mathbf{k}}^* + (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) \cdot (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}^*) \right\}.$$

From (52.3), we have

$$(\mathbf{k} \times \mathbf{A}_{\mathbf{k}}) \cdot (\mathbf{k} \times \mathbf{A}_{\mathbf{k}}^*) = k^2 \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}^*,$$

so that

$$\mathscr{E} = \frac{V}{8\pi c^2} \sum_{\mathbf{k}} \{ \dot{\mathbf{A}}_{\mathbf{k}} \cdot \dot{\mathbf{A}}_{\mathbf{k}}^* + k^2 c^2 \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{\mathbf{k}}^* \}. \tag{52.8}$$

Each term of this sum corresponds to one of the terms of the expansion (52.1).

Because of (52.4), the vectors $\mathbf{A}_{\mathbf{k}}$ are harmonic functions of the time with frequencies $\omega_{\mathbf{k}} = ck$, depending only on the absolute value of the wave vector. Depending on the choice of these functions, the terms in the expansion (52.1) can represent standing or running plane waves. We shall write the expansion so that its terms describe running waves. To do this we write it in the form

$$\mathbf{A} = \sum_{\mathbf{k}} (\mathbf{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \mathbf{a}_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r}}) \tag{52.9}$$

which explicitly exhibits that $\mathbf{A}$ is real, and each of the vectors $\mathbf{a}_{\mathbf{k}}$ depends on the time according to the law

$$\mathbf{a}_{\mathbf{k}} \sim e^{-i\omega_k t}, \quad \omega_{\mathbf{k}} = ck. \tag{52.10}$$

Then each individual term in the sum (52.9) will be a function only of the difference $\mathbf{k} \cdot \mathbf{r} - \omega_{\mathbf{k}} t$, which corresponds to a wave propagating in the $\mathbf{k}$ direction.

Comparing the expansions (52.9) and (52.1), we find that their coefficients are related by the formulas

$$\mathbf{A}_{\mathbf{k}} = \mathbf{a}_{\mathbf{k}} + \mathbf{a}_{-\mathbf{k}}^*,$$

and from (52.10) the time derivatives are related by

$$\dot{\mathbf{A}}_{\mathbf{k}} = -ick(\mathbf{a}_{\mathbf{k}} - \mathbf{a}_{-\mathbf{k}}^*).$$

Substituting in (52.8), we express the field energy in terms of the coefficients of the expansion (52.9). Terms with products of the form $\mathbf{a}_{\mathbf{k}} \cdot \mathbf{a}_{-\mathbf{k}}$ or $\mathbf{a}_{\mathbf{k}}^* \cdot \mathbf{a}_{-\mathbf{k}}^*$ cancel one another; also noting that the sums $\Sigma \mathbf{a}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^*$ and $\Sigma \mathbf{a}_{-\mathbf{k}} \mathbf{a}_{-\mathbf{k}}^*$ differ only in the labelling of the summation index, and therefore coincide, we finally obtain:

$$\mathscr{E} = \sum_{\mathbf{k}} \mathscr{E}_{\mathbf{k}}, \quad \mathscr{E}_{\mathbf{k}} = \frac{k^2 V}{2\pi} \mathbf{a}_{\mathbf{k}} \cdot \mathbf{a}_{\mathbf{k}}^*. \tag{52.11}$$

Thus the total energy of the field is expressed as a sum of the energies $\mathscr{E}_{\mathbf{k}}$, associated with each of the plane waves individually.

In a completely analogous fashion, we can calculate the total momentum of the field,

$$\frac{1}{c^2} \int \mathbf{S} dV = \frac{1}{4\pi c} \int \mathbf{E} \times \mathbf{H} dV,$$

for which we obtain

$$\sum_{\mathbf{k}} \frac{\mathbf{k}}{k} \frac{\mathscr{E}_{\mathbf{k}}}{c}. \tag{52.12}$$

This result could have been anticipated in view of the relation between the energy and momentum of a plane wave (see § 47).

The expansion (52.9) succeeds in expressing the field in terms of a series of discrete parameters (the vectors $\mathbf{a}_{\mathbf{k}}$), in place of the description in terms of a continuous series of parameters, which is essentially what is done when we give the potential $\mathbf{A}(x, y, z, t)$ at all points of space. We now make a transformation of the variables $\mathbf{a}_{\mathbf{k}}$, which has the result that the equations of the field take on a form similar to the canonical equations (Hamilton equations) of mechanics.

We introduce the real "canonical variables" $\mathbf{Q}_{\mathbf{k}}$ and $\mathbf{P}_{\mathbf{k}}$ according to the relations

$$\mathbf{Q}_{\mathbf{k}} = \sqrt{\frac{V}{4\pi c^2}} (\mathbf{a}_{\mathbf{k}} + \mathbf{a}_{\mathbf{k}}^*), \tag{52.13}$$

$$\mathbf{P}_{\mathbf{k}} = -i\omega_k \sqrt{\frac{V}{4\pi c^2}} (\mathbf{a}_{\mathbf{k}} - \mathbf{a}_{\mathbf{k}}^*) = \dot{\mathbf{Q}}_{\mathbf{k}}.$$

The Hamiltonian of the field is obtained by substituting these expressions in the energy (52.11):

$$\mathscr{H} = \sum_{\mathbf{k}} \mathscr{H}_{\mathbf{k}} = \sum_{\mathbf{k}} \tfrac{1}{2} (\mathbf{P}_{\mathbf{k}}^2 + \omega_k^2 \mathbf{Q}_{\mathbf{k}}^2). \tag{52.14}$$

Then the Hamilton equation $\partial \mathscr{H} / \partial \mathbf{P}_{\mathbf{k}} = \dot{\mathbf{Q}}_{\mathbf{k}}$ coincide with $\mathbf{P}_{\mathbf{k}} = \dot{\mathbf{Q}}_{\mathbf{k}}$, which is thus a consequence of the equations of motion. (This was achieved by an appropriate choice of the coefficient in (52.13).) The equations of motion, $\partial \mathscr{H} / \partial \mathbf{Q}_{\mathbf{k}} = -\dot{\mathbf{P}}_{\mathbf{k}}$, become the equations

$$\ddot{\mathbf{Q}}_{\mathbf{k}} + \omega_k^2 \mathbf{Q}_{\mathbf{k}} = 0, \tag{52.15}$$

that is, they are identical with the equations of the field.

Each of the vectors $\mathbf{Q}_{\mathbf{k}}$ and $\mathbf{P}_{\mathbf{k}}$ is perpendicular to the wave vector $\mathbf{k}$, i.e. has two independent components. The direction of these vectors determines the direction of polarization of the corresponding travelling wave. Denoting the two components of the vector $\mathbf{Q}_{\mathbf{k}}$ (in the plane perpendicular to $\mathbf{k}$) by $Q_{\mathbf{k}j}$, $j = 1, 2$, we have

$$\mathbf{Q}_{\mathbf{k}}^2 = \sum_j Q_{\mathbf{k}j}^2,$$

and similarly for $\mathbf{P_k}$. Then

$$\mathscr{H} = \sum_{\mathbf{k}j} \mathscr{H}_{\mathbf{k}j}, \quad \mathscr{H}_{\mathbf{k}j} = \tfrac{1}{2}(P_{\mathbf{k}j}^2 + \omega_k^2 Q_{\mathbf{k}j}^2). \tag{52.16}$$

We see that the Hamiltonian splits into a sum of independent terms $\mathscr{H}_{\mathbf{k}j}$, each of which contains only one pair of the quantities $Q_{\mathbf{k}j}$, $P_{\mathbf{k}j}$. Each such term corresponds to a travelling wave with a definite wave vector and polarization. The quantity $\mathscr{H}_{\mathbf{k}j}$ has the form of the Hamiltonian of a one-dimensional "oscillator", performing a simple harmonic vibration. For this reason, one sometimes refers to this result as the expansion of the field in terms of oscillators.

We give the formulas which express the field explicitly in terms of the variables $\mathbf{P_k}$, $\mathbf{Q_k}$. From (52.13), we have

$$\mathbf{a_k} = \frac{i}{k}\sqrt{\frac{\pi}{V}}\,(\mathbf{P_k} - i\omega_k \mathbf{Q_k}), \qquad \mathbf{a_k^*} = -\frac{i}{k}\sqrt{\frac{\pi}{V}}\,(\mathbf{P_k} + i\omega_k \mathbf{Q_k}). \tag{52.17}$$

Substituting these expressions in (52.1), we obtain for the vector potential of the field:

$$\mathbf{A} = 2\sqrt{\frac{\pi}{V}} \sum_{\mathbf{k}} \frac{1}{k}(ck\mathbf{Q_k} \cos \mathbf{k}\cdot\mathbf{r} - \mathbf{P_k} \sin \mathbf{k}\cdot\mathbf{r}). \tag{52.18}$$

For the electric and magnetic fields, we find

$$\mathbf{E} = -2\sqrt{\frac{\pi}{V}} \sum_{\mathbf{k}}(ck\mathbf{Q_k} \sin \mathbf{k}\cdot\mathbf{r} + \mathbf{P_k} \cos \mathbf{k}\cdot\mathbf{r}),$$

$$\mathbf{H} = -2\sqrt{\frac{\pi}{V}} \sum_{\mathbf{k}} \frac{1}{k}\{ck(\mathbf{k}\times\mathbf{Q_k}) \sin \mathbf{k}\cdot\mathbf{r} + (\mathbf{k}\times\mathbf{P_k}) \cos \mathbf{k}\cdot\mathbf{r}\}. \tag{52.19}$$