ESERCIZIO 1

2D Fermions, $S=\frac{1}{2}$, $\pi=0$, relativistic

$$\epsilon_{\bar{p},S_z} = \epsilon_{\bar{p}} = \sqrt{(pc)^2+(m_0c^2)^2} - m_0c^2$$

1) $p_F$?
$$2\,\frac{\pi p_F^2}{h^2}A = N \Rightarrow p_F^2 = \frac{h^2}{2\pi}\frac{N}{A}$$

2) $E$?

$$E = 2\sum_{\bar{p}} \epsilon_p = 2\int \frac{d\bar{p}}{h^2/A}\left\{\sqrt{(pc)^2+(m_0c^2)^2} - m_0c^2\right\}$$

$$= \frac{2\cdot 2\pi A}{h^2}\int_0^{p_F} dp\, p\left\{\sqrt{(pc)^2+(m_0c^2)^2} - m_0c^2\right\}$$

$\boxed{y=(pc)^2}$

$$= \frac{2\pi A}{(hc)^2}\int_0^{(p_Fc)^2} dy\left\{\sqrt{y+(m_0c^2)^2} - m_0c^2\right\}$$

$$= \frac{2\pi A}{(hc)^2}\left\{\frac{2}{3}\left[y+(m_0c^2)^2\right]^{3/2} - m_0c^2 y\right\}_0^{(p_Fc)^2}$$

$$E = \frac{2\pi A}{(hc)^2}\left\{\frac{2}{3}\left[(p_Fc)^2+(m_0c^2)^2\right]^{3/2} - m_0c^2(p_Fc)^2 - \frac{2}{3}(m_0c^2)^3\right\}$$

3) i) $p_F c \ll m_0 c^2$

$$E = \frac{2\pi A}{(hc)^2}\left\{ \frac{2}{3}(m_0 c^2)^3\left(\left[1+\left(\frac{p_F c}{m_0 c^2}\right)^2\right]^{3/2} - 1\right) - m_0 c^2 (p_F c)^2\right\}$$

$$\simeq \frac{2\pi A}{(hc)^2}\left\{ \frac{2}{3}(m_0 c^2)^3\left(\cancel{1} + \frac{3}{2}\left(\frac{p_F c}{m_0 c^2}\right)^2 + \frac{3}{8}\left(\frac{p_F c}{m_0 c^2}\right)^4 - \cancel{1}\right) - m_0 c^2 (p_F c)^2\right\}$$

$$E \simeq \frac{2\pi A}{(hc)^2}\left\{ \cancel{(p_F c)^2 m_0 c^2} + \frac{1}{4}\frac{(p_F c)^4}{m_0 c^2} - \cancel{m_0 c^2 (p_F c)^2}\right\}$$

$$\boxed{E \simeq \frac{\pi}{(hc)^2}\frac{1}{2}\frac{(p_F c)^4}{m_0 c^2} A} = N \frac{1}{2}\frac{p_F^2}{2 m_0}$$

ii) $m_0 c^2 \ll p_F c$

$$E = \frac{2\pi A}{(hc)^2}\left\{ \frac{2}{3}(p_F c)^3\left[1+\left(\frac{m_0 c^2}{p_F c}\right)^2\right]^{3/2} - m_0 c^2 (p_F c)^2 - \frac{2}{3}(m_0 c^2)^3\right\}$$

$$\simeq \frac{2\pi A}{(hc)^2}\left\{ \frac{2}{3}(p_F c)^3\left(1 + \frac{3}{2}\left(\frac{m_0 c^2}{p_F c}\right)^2 - \left(\frac{m_0 c^2}{p_F c}\right)^3 - \frac{3}{2}\left(\frac{m_0 c^2}{(p_F c)}\right)\right)\right\}$$

$$\boxed{E \simeq \frac{4\pi A}{3(hc)^2}(p_F c)^3} = \frac{2}{3} N \cdot p_F c$$

4) $(p_F c)^2 = \frac{h^2}{2\pi} c^2 \frac{N}{A} \equiv \frac{\gamma}{A}$

c) $$P = -\frac{\partial E}{\partial A} = -\frac{\partial}{\partial A}\left\{\frac{1}{2}\frac{\pi}{(hc)^2}\frac{(p_F c)^4}{m_0 c^2} A\right\}$$

$$= -\frac{\partial}{\partial A}\left\{\frac{\pi}{2(hc)^2}\frac{\gamma^2}{m_0 c^2 A}\right\}$$

$$= \frac{\pi}{2(hc)^2}\frac{\gamma^2}{A^2}\frac{1}{m_0 c^2} = \frac{\pi}{2(hc)^2}\frac{(p_F c)^4}{m_0 c^2}$$

$$\Longrightarrow \boxed{P = \frac{E}{A}}$$

u) $$P = -\frac{\partial}{\partial A}\left\{\frac{4\pi A}{3(hc)^2}\frac{\gamma^{3/2}}{A^{3/2}}\right\} = \frac{1}{2}\frac{4\pi A}{3(hc)^2}\left(\frac{\gamma}{A}\right)^{3/2}$$

$$\boxed{P = \frac{1}{2}\frac{E}{A}}$$

ESERCIZIO 2

① $$g(\epsilon) = \frac{1}{A}\sum_{\vec{p}} \delta\left(\epsilon - \frac{p_x^2}{2m_x} - \frac{p_y^2}{2m_y}\right)$$

$$\boxed{\begin{array}{l} p_x = \sqrt{2m_x}\, t_x \\ p_y = \sqrt{2m_y}\, t_y \end{array}}$$

$$= \frac{1}{\cancel{A}} \int \frac{dp_x\, dp_y}{\frac{h^2}{\cancel{A}}}\, \delta\left(\epsilon - \frac{p_x^2}{2m_x} - \frac{p_y^2}{2m_y}\right)$$

$$= \frac{2\sqrt{m_x m_y}}{h^2} \int dt_x\, dt_y\, \delta(\epsilon - t_x^2 - t_y^2)$$

$$\boxed{m^* = \sqrt{m_x m_y}}$$

$$= \frac{2\sqrt{m_x m_y}}{h^2} \int d\vec{t}\, \delta(\epsilon - t^2)$$

$$= \frac{2 m^*}{h^2}\, 2\pi \int dt\, t\, \delta(\epsilon - t^2) \qquad s = t^2$$

$$= \frac{2\pi m^*}{h^2} \int_0^\infty ds\, \delta(\epsilon - s)$$

$$\boxed{g(\epsilon) = \frac{2\pi m^*}{h^2}\, \theta(\epsilon) \qquad m^* = \sqrt{m_x m_y}}$$

② $$\frac{\ln Z}{A} = -\frac{1}{A}\sum_{\vec{p}} \ln\left(1 - e^{-\beta\epsilon_p} z\right)$$

$$= -\int d\epsilon\, g(\epsilon) \ln\left(1 - z e^{-\beta\epsilon}\right)$$

$$= -\frac{2\pi m^*}{h^2} \int_0^\infty d\epsilon\, \ln\left(1 - z e^{-\beta\epsilon}\right)$$

$$\frac{\ln Z}{A} = -\frac{2\pi m^* k_B T}{h^2}\int_0^\infty ds\, \ln(1-ze^{-s}) \qquad \boxed{s=\beta\varepsilon}$$

$$= \frac{1}{\lambda_T^2}\int_0^\infty ds\, \left[-\ln(1-ze^{-s})\right] \qquad \boxed{\frac{1}{\lambda_T^2} = \frac{2\pi m^* k_B T}{h^2}}$$

$$= \frac{1}{\lambda_T^2}\int_0^\infty ds \sum_{n=1}^\infty \frac{z^n e^{-sn}}{n} \qquad \boxed{t=sn}$$

$$= \frac{1}{\lambda_T^2}\sum_{n=1}^\infty \frac{z^n}{n^2}\int_0^\infty dt\, e^{-t}$$

$$= \frac{1}{\lambda_T^2}\sum_{n=1}^\infty \frac{z^n}{n^2} = \frac{g_2(z)}{\lambda_T^2}$$

$$\boxed{\beta P = \frac{\ln Z}{A} = \frac{g_2(z)}{\lambda_T^2}}$$

③ $$E = -\left.\frac{\partial \ln Z}{\partial \beta}\right|_{z,A} = -\frac{\partial}{\partial\beta}\frac{A g_2(z)}{\lambda_T^2}$$

$$= A g_2(z)\left(-\frac{\partial}{\partial\beta}\frac{1}{\lambda_T^2}\right) = A g_2(z)\frac{1}{\beta\lambda_T^2}$$

$$\boxed{E = k_B T\frac{g_2(z)}{\lambda_T^2}A = P\cdot A}$$

(4) Per chiarezza qui indichiamo l'energia libera di Helmoltz con la lettera F

$$F = G - \mathcal{P}A = \langle N \rangle \mu - PA$$

$$\langle N \rangle = z \frac{\partial \ln Z}{\partial z}\bigg|_{A,T} = z \frac{\partial}{\partial z}\left[A \frac{g_2(z)}{\lambda_T^2}\right]$$

$$\langle N \rangle = A \frac{g_1(z)}{\lambda_T^2}$$

$$F = A \frac{g_1(z)}{\lambda_T^2} k_B T \ln z - A k_B T \frac{g_2(z)}{\lambda_T^2}$$

$$\frac{S}{k_B} = \frac{1}{k_B} \frac{E - F}{T} = \beta E - \beta F$$

$$\frac{S}{k_B} = \frac{g_2(z)}{\lambda_T^2} A - \frac{g_1(z)}{\lambda_T^2} \ln z A + \frac{g_2(z)}{\lambda_T^2} A$$

$$\boxed{\frac{S}{k_B} = \left[2 \frac{g_2(z)}{\lambda_T^2} - \frac{g_1(z)}{\lambda_T^2} \ln z\right] A}$$