# ESERCIZIO 1

(1) $\bar{F}(\bar{r}) = e\bar{E}(\bar{r})$ $\qquad \bar{E}(\bar{r}) = -e\left[\rho \frac{4\pi}{3} r^3\right]\frac{\bar{r}}{r^3}$

$$\bar{F}(\bar{r}) = -e^2 \frac{1}{\cancel{\frac{4\pi}{3}} R^3} \cancel{\frac{4\pi}{3}} \cancel{r^3} \frac{\bar{r}}{\cancel{r^3}} = -\frac{e^2 \bar{r}}{R^3}$$

$$\boxed{\bar{F}(\bar{r}) = -\frac{e^2 \bar{r}}{R^3}}$$

(2) $M\ddot{\bar{r}} = -\frac{e^2 \bar{r}}{R^3} \Rightarrow \bar{r}(t) = \bar{r}_0 e^{-i\omega t}$

$$\Rightarrow -M\bar{r}_0 \omega^2 = -\frac{e^2 \bar{r}_0}{R^3}$$

$$\boxed{\omega^2 = \frac{e^2}{MR^3}} \boxed{= \Omega_{IS}^2}$$

(3) $M = 3.82 \cdot 10^{-23}\, g$ [atomic mass for Na is 23.0 g]

$R = 2.08 \cdot 10^{-8}$ cm, AM, Tab 1.1

$$\Omega_{IS}^2 = \frac{(4.80 \cdot 10^{-10})^2}{3.82 \cdot 10^{-23} (2.08 \cdot 10^{-8})^3} = 6.70 \cdot 10^{26}$$

$$\boxed{\Omega_{IS} = 2.59 \cdot 10^{13}\ s^{-1}}$$

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- Electrons $\quad \omega_p^2 = \frac{4\pi}{m_e} \rho e^2$

- Ions $\quad \Omega_p^2 = \frac{4\pi \rho e^2}{M} = \frac{4\pi}{M} \frac{1}{\frac{4\pi}{3} R^3} e^2$

$$\Omega_p^2 = \frac{3e^2}{MR^3} = 3\,\Omega_{IS}^2$$

$$\boxed{\Omega_p = \sqrt{3}\,\Omega_{IS}}$$

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$$\Omega_p^2 = \frac{4\pi \rho e^2}{M} \Rightarrow \frac{4\pi \rho e^2}{M} \frac{1}{\epsilon(\kappa)} = \tilde{\Omega}_p^2(\kappa)$$

$$\epsilon(\kappa) = 1 + \frac{\varkappa^2}{\kappa^2}$$

$$\tilde{\Omega}_p^2(\kappa) = \frac{\Omega_p^2}{\epsilon(\kappa)} = \frac{\Omega_p^2}{1 + \frac{\varkappa^2}{\kappa^2}} \simeq \frac{\Omega_p^2 \kappa^2}{\varkappa^2}$$

$$\tilde{\Omega}_p(\kappa) \simeq \Omega_p \frac{\kappa}{\varkappa} \equiv c_s \kappa$$

$$c_s = \frac{\Omega_p}{\varkappa}$$

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$$(17.50)\ \text{AM} \Rightarrow \varkappa = \kappa_0 = \frac{2.95}{(r_s/a_0)^{\frac{1}{2}}}\ \text{Å}^{-1}$$

Task 1.1

$$k_0 = \frac{2.95}{\sqrt{3.93}} \text{ Å}^{-1}$$

$$k_0 = 1.49 \cdot 10^{8} \text{ cm}^{-1}$$

$$c_s = \frac{\Omega_p}{k_0} = \frac{\sqrt{3}\,\Omega_{IS}}{k_0} = \frac{\sqrt{3} \cdot 2.59 \cdot 10^{13}}{1.49 \cdot 10^{8}} \text{ cm s}^{-1}$$

$$c_s = 3.01 \cdot 10^{5} \text{ cm s}^{-1} = 3.01 \cdot 10^{3} \text{ m s}^{-1}$$

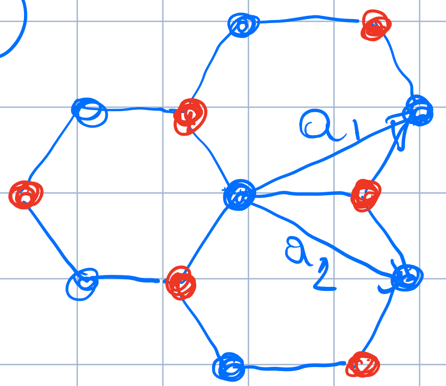
$$\boxed{c_s = 3.01 \cdot 10^{3} \text{ m/s}}$$

ESERCIZIO 2

$\bar R = l\bar a_1 + m\bar a_2$, $l,m \in \mathbb{Z}$

• , A ; • , B

(1)




Ogni sito • ha 3 primi vicini • alle posizioni

$\vec b_2$, $\vec b_2 - \bar a_2$, $\vec b_2 - \bar a_1$

$$\hat{\mathcal{H}} = |J| \sum_{\bar R} \hat{\vec S}(\bar R) \cdot \left[ \hat{\vec S}(\bar R + \bar b_2) + \hat{\vec S}(R + b_2 - a_2) \right.$$
$$\left. + \hat{\vec S}(\bar R + \bar b_2 - \bar a_1) \right] - \sum_{\bar R} \mu_B g h \left[ \hat S_z(\bar R) + \hat S_z(\bar R + \bar b_2) \right]$$

$\bar H = (0, 0, h)$

$\boxed{\text{p.v. } \bar d_1 = \bar b_2,\ \bar d_2 = \bar b_2 - \bar a_2,\ \bar d_3 = \bar b_2 - \bar a_1}$

(2)

$$\hat{\mathcal{H}} = -\mu_B g h \sum_{\bar R} \left[ \hat S_z(\bar R) + \hat S_z(\bar R + \bar b_2) \right]$$

$\boxed{S_z = \pm 1/2}$

$$E(\{S_z(\bar R)\}, \{S_z(\bar R + \bar b_2)\}) =$$
$$-\mu_B h \sum_{\bar R} \left[ \sigma(\bar R) + \sigma(\bar R + \bar b_2) \right]$$

$$Z = \sum_{\{\sigma(\bar R)\}\ \{\sigma(\bar R + \bar b_2)\}} e^{+\beta \mu_B h \sum_{\bar R} (\sigma(\bar R) + \sigma(\bar R + \bar b_2))}$$

$\boxed{\sigma = \pm 1}$

$$Z = \sum_{\substack{\{\sigma(\bar{R})\} \\ \{\sigma(\bar{R}+\bar{b}_2)\}}} \prod_{R} e^{\beta\mu_B h\sigma(\bar{R})} \cdot e^{\beta\mu_B h\sigma(\bar{R}+\bar{b}_2)}$$

$$= \prod_{R} \sum_{\sigma(\bar{R})=\pm 1} \sum_{\sigma(\bar{R}+\bar{b}_2)=\pm 1} e^{\beta\mu_B\sigma(\bar{R})} e^{\beta\mu_B\sigma(\bar{R}+\bar{b}_2)}$$

$$Z = 4 \prod_{\bar{R}} \cosh(\beta\mu_B h)\cosh(\beta\mu_B h)$$

$$Z = 4\left[(\cosh(\beta\mu_B h))^2\right]^{N/2} = [2\cosh\beta\mu_B h]^N$$

$N$ : number of sites in the crystal

$N/2$ : number of sites of the triangular crystal

$$F = -k_B T \ln Z = -k_B T\left[N\ln(2\cosh(\beta\mu_B h))\right]$$

$$M = -\frac{1}{A}\frac{\partial F}{\partial h} = \frac{N k_B T}{A}\beta\mu_B \,\mathrm{tgh}(\beta\mu_B h)$$

$$\boxed{M = \frac{N\mu_B}{A}\,\mathrm{tgh}(\beta\mu_B h)}$$

The result above can be also obtained by looking at the average magnetization of one site, as sites are independent!

③ We calculate the average magnetic field at • and at •

•) $-\mu_B g\, \hat{\bar{S}}(\bar{R})\cdot\bar{H} + |J|\hat{\bar{S}}(\bar{R}) \sum_{i=1}^{3} \hat{\bar{S}}(\bar{R}+\bar{d}_i)$

$$= -\mu_B g\, \hat{\bar{S}}(\bar{R}) \left[ \bar{H} - \frac{|J|}{\mu_B g} \sum_{i=1}^{3} \hat{\bar{S}}(\bar{R}+d_i) \right]$$

$$\langle \hat{\bar{S}}(\bar{R}+\bar{d}_i) \rangle = S_B \hat{z}$$

$$M_B = \frac{N}{2A} \mu_B g\, S_B \Rightarrow S_B = \frac{M_B}{\left(\frac{N\mu_B}{A}\right)}$$

Finally

$$-\mu_B g\, \hat{S}_z(R) \left[ h - \frac{3|J|}{\mu_B g} \cdot S_B \right]$$

$$= -\mu_B g\, \hat{S}_z(\bar{R}) \left[ h - \lambda M_B \right]$$

$$= -\mu_B g\, \hat{S}_z(\bar{R})\, h_A$$

$$\boxed{h_A = h - \lambda M_B} \; ; \; \lambda = \frac{3|J|}{\mu_B g \frac{N\mu_B}{A}} = \frac{3|J|}{\frac{2N}{A}\mu_B^2}$$

•) In modo analogo si ottiene

$$h_B = h - \lambda M_A$$

(4) For independent spins the magnetization per site is $\mu_B \,\mathrm{tgh}(\beta \mu_B h)$. Thus

$$M_A = \frac{N}{2A} \,\mathrm{tgh}\,(\beta \mu_B [h - \lambda M_B])$$

$$M_B = \frac{N}{2A} \,\mathrm{tgh}\,(\beta \mu_B [h - \lambda M_A])$$

(5) $h = 0$

$$(a) \left\{ \begin{array}{l} M_A = \frac{N}{2A} \,\mathrm{tgh}\,(-\beta \mu_B \lambda M_B) = -\frac{N}{2A} \,\mathrm{tgh}\,(\beta \mu_B \lambda M_B) \\ M_B = -\frac{N}{2A} \,\mathrm{tgh}\,(\beta \mu_B \lambda M_A) \end{array} \right.$$

(a) implies that $M_A$ and $M_B$ have opposite signs : $M_A M_B \leq 0$

$$\left. \begin{array}{l} M_A M_B = -\frac{N}{2A} M_B \,\mathrm{tgh}\,(\beta \mu_B \lambda M_B) \\ \quad = -\frac{N}{2A} M_A \,\mathrm{tgh}\,(\beta \mu_B \lambda M_A) \end{array} \right\} (b)$$

Let's set $y_{A,B} = \beta \mu_B \lambda M_{A,B}$

(b) implies $y_A \,\mathrm{tgh}(y_A) = y_B \,\mathrm{tgh}(y_B)$,

or $y_A = -y_B$, as $M_A M_B \leq 0$,

$M_B = -M_A$

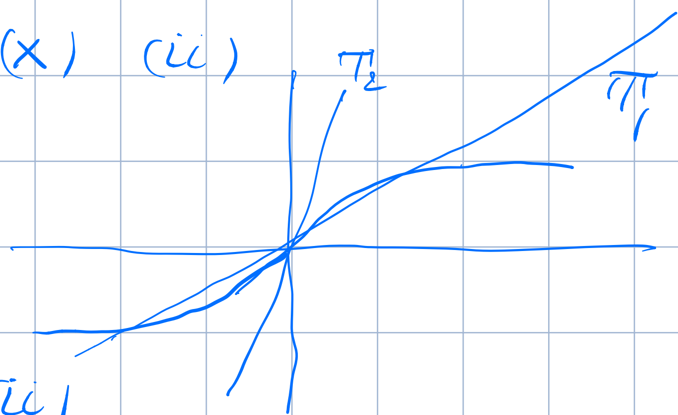
We conclude that

$$M_A = \frac{N}{2A}\mu_B \, \text{tgh}\left(\beta\mu_B\lambda M_A\right)$$

which can be solved setting

$$x = \beta\mu_B\lambda M_A \; :$$

$$\begin{cases} M_A = \dfrac{k_B T}{\mu_B\lambda}\, x & \text{(i)} \\ M_A = \dfrac{N}{2A}\mu_B \, \text{tgh}(x) & \text{(ii)} \end{cases}$$




If $T_c$ is such that the slopes at the origin of (i) and (ii) are the same, for $T < T_c$ there will be a finite magnetization

$$\frac{k_B T_c}{\mu_B\lambda} = \frac{N}{2A}\mu_B \Rightarrow T_c = \frac{N}{2A}\mu_B^2 \frac{\lambda}{k_B}$$

$$T_c = \frac{\cancel{N}}{2\cancel{A}} \frac{\cancel{\mu_B^2}}{k_B} \frac{3|J|}{2\frac{\cancel{N}}{\cancel{A}}\cancel{\mu_B^2}} = \frac{3|J|}{4k_B}$$

$$\boxed{T_c = \frac{3|J|}{4k_B}}$$

6 $M = M_A + M_B = \frac{N}{2A}\mu_B \left\{ \mathrm{tgh}\left(\beta\mu_B[h-\lambda M_B]\right) + \mathrm{tgh}\left(\beta\mu_B[h-\lambda M_A]\right) \right\}$

$$\chi = \left.\frac{\partial M}{\partial h}\right|_{h=0}$$

as we are at $T > T_c$ and $M_A, M_B$ go to zero when $h \to 0$, we can linearize

$$M \simeq \frac{N}{2A}\mu_B \{\beta\mu_B\}\{2h - \lambda(M_A + M_B)\}$$

$$= \frac{N\mu_B^2 \beta h}{A} - \frac{N}{2A}\beta\mu_B^2\lambda M$$

$$\frac{\partial M}{\partial h} = \frac{N\mu_B^2}{A k_B T} - \frac{N\mu_B^2}{2A k_B T}\lambda\frac{\partial M}{\partial h}$$

$$= \chi = \frac{N\mu_B^2}{A k_B T} - \frac{N\mu_B^2}{2A k_B T}\lambda\chi$$

$$\chi\left[1 + \frac{N\mu_B^2\lambda}{2A k_B T}\right] = \frac{N\mu_B^2}{A k_B T} = \chi_0$$

$$\frac{N\mu_B^2}{2A k_B T}\lambda = \frac{\cancel{N\mu_B^2}}{2\cancel{A} k_B T}\,\frac{3|J|}{2\frac{\cancel{N}}{\cancel{A}}\cancel{\mu_B^2}} = \frac{3|J|}{4 k_B T} = \frac{T_c}{T}$$

$$\chi = \frac{\chi_0}{1 + \frac{T_c}{T}} \simeq \frac{\chi_0 T_c}{T + T_c}, \quad T \gtrsim T_c$$