Condensed Matter Physics II. – A.A. 2019-2020, June 12 2020

(time 3 hours)

Solve the following two exercises, each has a maximum score of 18 for a total of 36. A score between 33 e 36 corresponds to 30 cum laude, between 30 e 32 is renormalized to 30 (the maximum official score, without laude).

NOTE:

- Give all details which help in understanding the proposed solution. Answers which only contain the final result or not enough detail will be judged insufficient and discarded;
- If you are requested to give evaluation/estimates, do so using 3 significant figures.

Exercise 1: Longwavelength phonon frequencies in a lattice with long range interactions

Consider the small oscillations of atoms in a monoatomic Bravais lattice in dimension d.

1. Obtain an expression for the square of the sound velocity assuming that for small wavevectors k you can expand to leading order the trigonometric function in the expression for the squared phonon frequency in terms of the dynamical matrix.
2. Let's assume now that the atoms interact with the pair potential $\phi(|\mathbf{r}|)$, so that the dynamical matrix is
$$D_{\mu\nu}(\mathbf{R}-\mathbf{R}') = \delta_{\mathbf{R},\mathbf{R}'} \sum_{\mathbf{R}''(\neq\mathbf{R})} \phi_{\mu\nu}(\mathbf{R}-\mathbf{R}'') - (1-\delta_{\mathbf{R},\mathbf{R}'})\phi_{\mu\nu}(\mathbf{R}-\mathbf{R}').$$
Focussing on the large distance behavior, we assume that $\phi(r) \approx C/r^\alpha$ with $\alpha > 0$. From the equation above and $\phi_{\mu\nu}(r) = \partial^2\phi/\partial r_\mu \partial r_\nu$ it follows that
$$D_{\mu\nu}(\mathbf{R}) \approx A/R^{\alpha+2}. \tag{1}$$
Use such a behavior to find a sufficient condition on α in dimension d that yields a finite sound velocity.
3. To study better the dependence of phonon frequencies at small k rewrite the sum over $\mathbf{R}$ in eq. (22.59, AM) as an integral over $\mathbf{R}$ with lower limit a_L the lattice parameter of the Bravais.
4. In the above integral change from the integration variable R to $y = kR/2$, neglecting at the same time the angular dependence in the trigonometric function.
5. Find what is the range of α values that yield a finite frequency. Such a range will depend on d.
6. If at given d you find $\alpha_m < \alpha < \alpha_M$ you should have realized that α_M is related to the behavior of the integral on y around the lower limit. Thus, you should be able to find a logarithmic contribution to $\omega^2(k)$ for small k as in the exercise 1, page 448, AM.

Esercizio 2: Meissner effect in a superconducting cylinder

Consider an infinite superconducting cylinder of radius R, i.e., $0 \leq \sqrt{x^2+y^2} \leq R$ in an external uniform magnetic field $\mathbf{H} = H_0 \hat{z}$.

1. Write down the equation obeyed by $\mathbf{B}$ inside the cylinder.

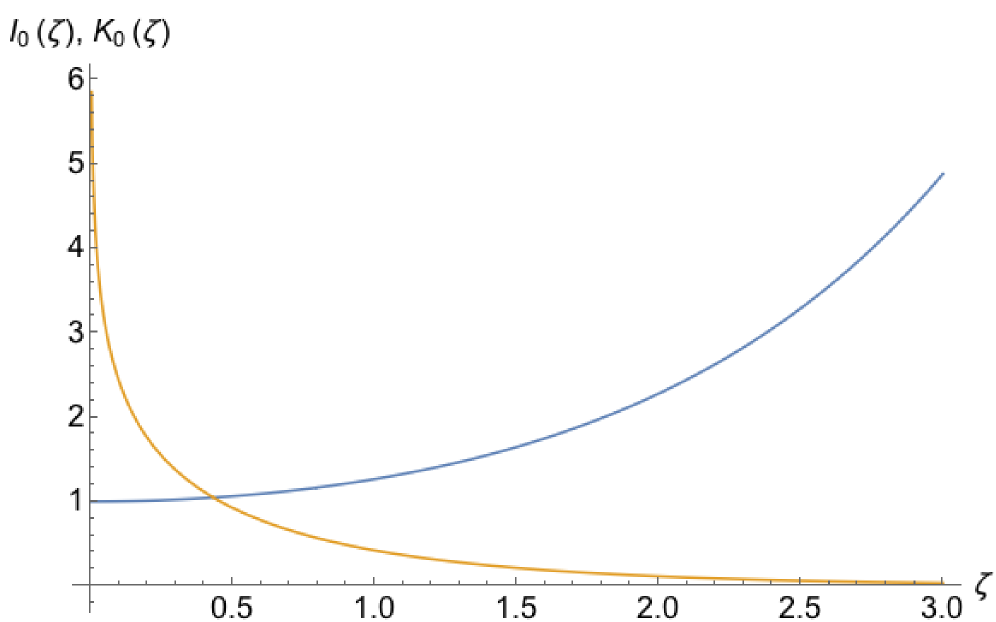
2. Given the symmetry of the problem one may assume that $\mathbf{B}$ only depends on the distance $\rho = \sqrt{x^2+y^2}$ from the axis of the cylinder, thus it is natural to rewrite the equation for $\mathbf{B}(\rho)$ in cylindrical coordinates.

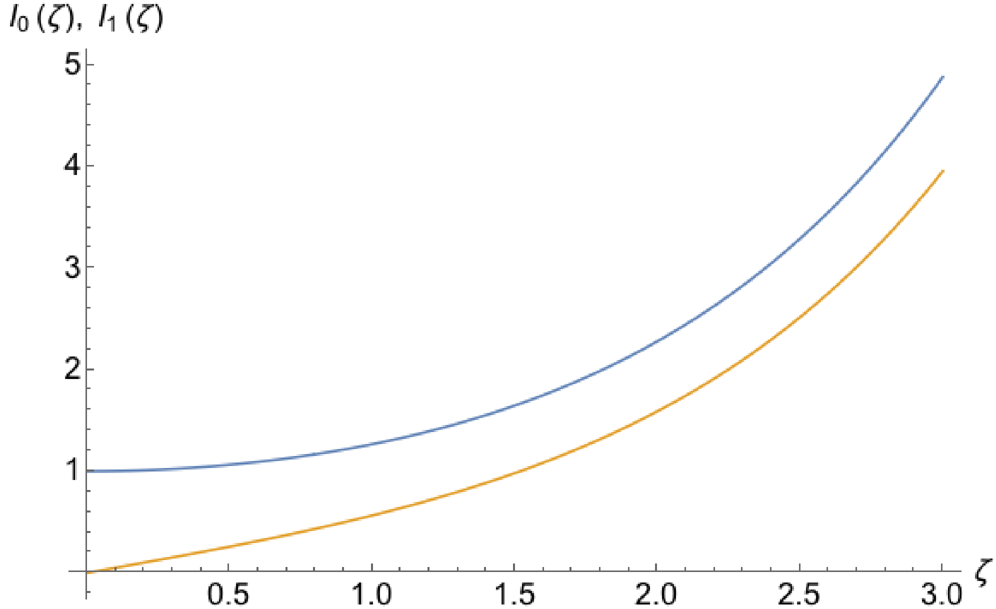
3. Find the relation between the equation above and that obeyed by the modified Bessel functions w
$$\zeta^2 \frac{d^2 w}{d\zeta^2} + \zeta \frac{dw}{d\zeta} - (\zeta^2 - \nu^2) w = 0,$$
with solutions for $\nu = 0$ $I_0(\zeta)$ and $K_0(\zeta)$.

4. Express $\mathbf{B}$ in terms of I_0 and K_0.

5. Using the fact that (i) $K_0(\zeta) \approx -\log \zeta$ as $\zeta \to 0$ and $I_0(\zeta)$ grows as e^ζ for ζ large and (ii) the boundary conditions on $\mathbf{B}$ at $\rho = R$ determine in full $\mathbf{B}(\rho)$.

6. Obtain $\mathbf{j}(\rho)$ inside the cylinder. Note: $I_0'(\zeta) = I_1(\zeta)$ and $I_1(\zeta)$ grows as e^ζ for ζ large.





$I_0(\zeta),\ I_1(\zeta)$
5
4
3
2
1
0.5
1.0
1.5
2.0
2.5
3.0
ζ