① $D(\vec{k}) = -2 \sum_{\vec{R}} D(\vec{R}) \sin^2 \frac{\vec{k}\cdot\vec{R}}{2} \simeq -\frac{1}{2} \sum_{\vec{R}} D(\vec{R}) (\vec{k}\cdot\vec{R})^2$

$= -\frac{k^2}{2} \sum_{\vec{R}} D(\vec{R}) (\hat{k}\cdot\vec{R})^2 = k^2 \tilde{D}(\hat{k})$

$\tilde{D}(\hat{k}) = -\frac{1}{2} \sum_{\vec{R}} D(\vec{R}) (\hat{k}\cdot\vec{R})^2$ $\quad \boxed{D, \tilde{D} \text{ matrici } d \times d}$

$M\omega^2 \bar{\epsilon} = k^2 \tilde{D}(\hat{k}) \bar{\epsilon}$

$M\left(\frac{\omega}{k}\right)^2 \bar{\epsilon} = \tilde{D}(\hat{k}) \bar{\epsilon} \quad \Rightarrow \quad \left(\frac{\omega}{k}\right)^2 = c_s(\hat{k})^2$

$M c_s^2(\hat{k}) = \lambda_s(\hat{k})$; $\lambda_s(\hat{k})$: autovalori di $\tilde{D}(\hat{k})$
$s = 1, .., d$

② $\tilde{D}(\hat{k}) = -\frac{1}{2} \sum_{\vec{R}} D(\vec{R}) (\hat{k}\cdot\vec{R})^2 \simeq -\frac{1}{2} \int \frac{d\vec{R}}{v_{ws}} D(\vec{R}) (\hat{k}\cdot\vec{R})^2$

$\simeq -\frac{1}{2} \int \frac{dR\, R^{d-1}}{v_{ws}} \frac{A}{R^{\alpha+2}} \int d\Omega_d\, R^2 \cos^2\theta_d$ *

$= -\frac{A}{2} C \int \frac{dR}{v_{ws}} \frac{R^{d+1}}{R^{\alpha+2}} \simeq -\frac{A}{2} C \int_{a_L}^{\infty} \frac{dR\, R^{d-1-\alpha}}{a_L^d}$

$= -\frac{A\,C}{2\, a_L^d} \frac{1}{d-\alpha} \left[R^{d-\alpha} \right]_{a_L}^{\infty}$

- Affinché $\tilde{D}(\hat{k}) < \infty$ $\quad d - \alpha < 0 \quad \Rightarrow \boxed{\alpha > d}$

* $C = \int d\Omega_d \cos^2\theta_d$, $\cos\theta_d = \vec{R}\cdot\hat{k}/R$

③+④ $$D(\bar{k}) = -2\sum_{\bar{R}} D(\bar{R})\sin^2\frac{\bar{k}\cdot\bar{R}}{2} \simeq -2\int\frac{d\bar{R}}{v_{ws}} D(\bar{R})\sin^2\frac{\bar{k}\cdot\bar{R}}{2}$$

$$\simeq -2\int_{a_L}^{\infty}\frac{dR\, R^{d-1}}{v_{ws}}\int d\Omega_d\, \frac{A}{R^{\alpha+2}}\sin^2\left(\frac{kR\cos\theta_d}{2}\right)$$

$$\simeq -2\int_{ka_L/2}^{\infty} dy\; y^{d-\alpha-3}\left(\frac{2}{k}\right)^{d-\alpha-2} A\sin^2 y$$

Per ottenere l'ultima eguaglianza abbiamo trascurato la dipendenza angolare!

$$D(\bar{k}) \simeq -2A\left(\frac{2}{k}\right)^{d-\alpha-2}\int_{ka_L/2}^{\infty} dy\,\frac{\sin^2 y}{y^{\alpha+3-d}}$$

⑤ $D(\bar{k})$ finito richiede

$$1 < \alpha+3-d < 3$$

convergenza ad ∞ — convergenza a 0 $(qa_L/2 \to 0)$

Ovvero

$$\alpha_m = d-2 < \alpha < d = \alpha_M$$

$$D(\bar{k}) \simeq -2A\left(\frac{k}{2}\right)^{\alpha+2-d} I\left(\frac{k a_L}{2}\right)$$

$$\Rightarrow \omega^2 \sim k^{\alpha+2-d}, \quad \omega \sim k^{(\alpha+2-d)/2}$$

$0 < (\alpha + 2 - 4)/2 < 1$

Al variare di α dal minimo al massimo per piccoli k ω passa da $\sim k^0 =$ cost a k, andamento acustico!

⑥ $\alpha = d$

$$D(\vec{k}) \sim -2A\left(\frac{k}{2}\right)^2 \int_{\frac{ka_L}{2}}^{\infty} dy \, \frac{\sin^2 y}{y^3}$$

- Andamento a ∞ ok.
- Divergenza al limite inferiore quando $\frac{ka_L}{2} \to 0$

- Dividiamo l'integrale in due parti: una da $k_0 a_L/2$ a ∞, regolare; più $\int_{\frac{ka_L}{2}}^{k_0 a_L/2} dy \, \frac{\sin^2 y}{y^3}$

Quindi $I(ka_L/2) = \int_{\frac{ka_L}{2}}^{k_0 a_L/2} dy \, \frac{\sin^2 y}{y^3} + \int_{k_0 a_L/2}^{\infty} dy \, \frac{\sin^2 y}{y^3}$

$= I_1(ka_L/2) + I_2$

Scegliamo $k_0 a_L/2 \ll 1$ ma finito,
quindi
$I_2 < \infty$

Invece $I_1\left(\frac{k a_L}{2}\right) \simeq \int_{\frac{k a_L}{2}}^{k_0 a_L/2} dy \, \frac{y^2}{y^3} = \ln \frac{k_0}{k} = -\ln \frac{k}{k_0}$

Per $k \to 0$ I_1 domina su I_2
e quindi
$D(\vec{k}) \simeq \frac{A}{2} k^2 \ln \frac{k}{k_0}$

$\Rightarrow \quad \omega \sim k \sqrt{|\ln(k/k_0)|}$

ESERCIZIO 2

① $$\nabla^2 \bar{B} = \frac{\bar{B}}{\lambda^2} \qquad \bar{B} = (B_x, B_y, B_z)$$

② $$B_\alpha''(\rho) + \frac{B_\alpha'(\rho)}{\rho} = \frac{B_\alpha}{\lambda^2}$$

③ $$\lambda^2 B_\alpha''(\rho) + \frac{\lambda}{\rho} \lambda B_\alpha'(\rho) = B_\alpha$$

Poniamo $\xi = \frac{\rho}{\lambda} \Rightarrow \frac{d}{d\rho} = \frac{1}{\lambda}\frac{d}{d\xi}$

Quindi
$$\lambda^2 \cdot \frac{1}{\lambda^2}\frac{d^2 B_\alpha}{d\xi^2} + \frac{1}{\xi}\lambda \frac{1}{\lambda}\frac{dB_\alpha}{d\xi} = B_\alpha$$

$$B_\alpha''(\xi) + \frac{1}{\xi} B_\alpha'(\xi) = B_\alpha$$

$$\Rightarrow \quad \xi^2 B_\alpha''(\xi) + \xi B_\alpha'(\xi) - \xi^2 B_\alpha = 0$$

Soluzioni per $B_\alpha(\xi)$ sono $I_0(\xi), K_0(\xi)$

④ $$B_\alpha\left(\frac{\rho}{\lambda}\right) = A_\alpha I_0\left(\frac{\rho}{\lambda}\right) + C_\alpha K_0\left(\frac{\rho}{\lambda}\right)$$

(5) Poiché il superconduttore occupa tutto lo spazio $0 \leq \rho \leq R$ la soluzione $K_0\left(\frac{\rho}{\lambda}\right)$ è da scartare in quanto divergente

$$B_\alpha\left(\frac{\rho}{\lambda}\right) = A_\alpha I_0\left(\frac{\rho}{\lambda}\right)$$

Poiché il campo $\bar{B}$ deve essere continuo a $\rho = R$ e dall'esterno abbiamo $\bar{H} = (0, 0, H_0)$ otteniamo

$$A_x = A_y = 0 \quad e \quad A_z I_0\left(\frac{R}{\lambda}\right) = H_0$$

$$\bar{B} = \left(0, 0, H_0 \frac{I_0(\rho/\lambda)}{I_0(R/\lambda)}\right)$$

(6) Da $\nabla \times \bar{B} = \frac{4\pi}{c} \bar{j}$ otteniamo le componenti di $\bar{j}$

$$(\nabla \times B)_x = \partial_y B_z - \partial_z B_y = \partial_y B_z$$

$$\partial_y B_z = \frac{H_0}{I_0(R/\lambda)} \frac{1}{\lambda} \frac{\partial \rho}{\partial y} I_1\left(\frac{\rho}{\lambda}\right)$$

$$= \frac{H_0}{\lambda I_0\left(\frac{R}{\lambda}\right)} I_1\left(\frac{\rho}{\lambda}\right) \frac{y}{\rho}$$

$$(\nabla\times B)_y = \partial_z B_x - \partial_x B_z = -\partial_x B_z$$

$$= -\frac{H_0}{\lambda I_0(\frac{R}{\lambda})} I_1\left(\frac{\rho}{\lambda}\right)\frac{x}{\rho}$$

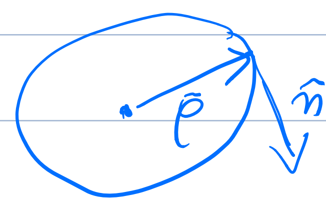
$$(\nabla\times B)_z = \partial_x B_y - \partial_y B_x = 0$$

Infine

$$\vec{J} = \frac{c}{4\pi}\frac{H_0}{\lambda I_0(\frac{R}{\lambda})}\frac{I_1(\frac{\rho}{\lambda})}{\rho}(y,-x,0)$$

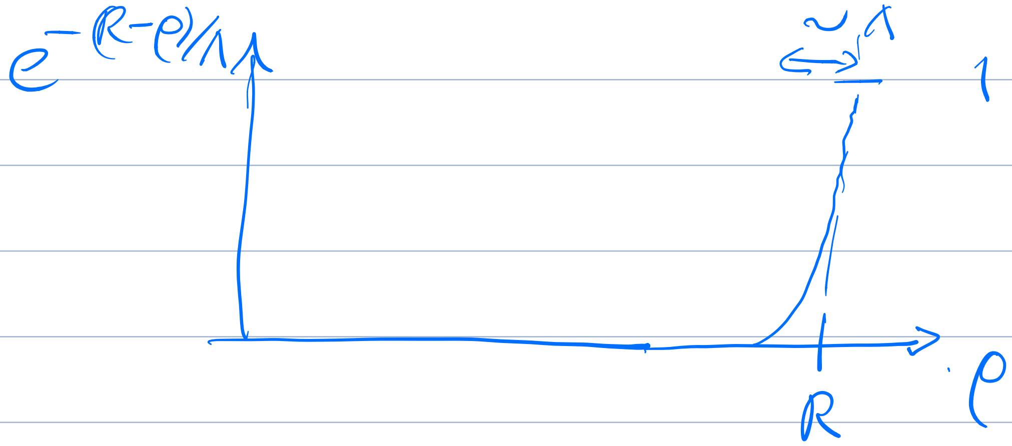
Il versore $\hat{n} = \frac{1}{\rho}(y,-x,0)$ é nel piano x,y ed é tangente alla superficie del cilindro

La corrente fluisce in senso orario.



Per un campione meso- o macro-scopico $R \gg \lambda$ e dato l'andamento della $I_0(\xi)$ a grandi argomenti avremo per $\rho \lesssim R$

$$J \sim \frac{I_1(\frac{\rho}{\lambda})}{I_0(\frac{R}{\lambda})} \sim e^{-(R-\rho)/\lambda}$$



$e^{-(R-\rho)/\Lambda}$
$\sim \Lambda$
1
R
ρ