I COMPITO - CMP-II 30.04.20 — ES. 1

(1) Gli autovalori sono crescenti in $|\vec{p}|$, quindi

3p

$$\sum_\sigma \sum_\tau \sum_{|\vec{p}|<p_F} = N = 2\cdot 2\cdot \frac{\pi p_F^2}{\left(\frac{h}{L}\right)^2} = \frac{4\pi p_F^2 A}{h^2} = N \Rightarrow$$

$$\Rightarrow p_F^2 = \frac{N}{A}\frac{h^2}{4\pi} = \frac{\pi N}{A}\hbar^2 \Rightarrow p_F = \hbar k_F \,, \quad k_F = \sqrt{\frac{\pi N}{A}} = k_F\sqrt{\pi n}$$

$$\boxed{p_F = \hbar\sqrt{\pi n}}$$

(2) $E = \sum_\sigma \sum_\tau \sum_{p<p_F} v^* p = 4v^* \sum_{p<p_F} p = 4v^* \int_0^{p_F} \frac{d\vec{p}}{\left(\frac{h}{L}\right)^2} p = \frac{4v^* A}{h^2} 2\pi \int_0^{p_F} dp\, p^2 = \frac{8\pi v^* A}{h^2}\frac{p_F^3}{3}$

3p

$$= v^* p_F \frac{8\pi A}{3h^2} p_F^2 = v^* p_F \frac{8\pi A}{3h^2}\frac{\pi N \hbar^2}{A} = v^* p_F \frac{1}{3}\frac{8\pi^2}{4\pi^2} N = N\frac{2}{3} v^* p_F$$

$$\boxed{\frac{E}{N} = \epsilon = \frac{2}{3} v^* p_F = \frac{2}{3}\epsilon_F}$$

(3) $E[n] = \int d\vec{r}\, n(\vec{r}) \frac{2}{3} v^* \hbar \sqrt{\pi\, n(\vec{r})} + \int d\vec{r}\, n(\vec{r}) v(r)$

2p

(4) $\frac{\delta E}{\delta n(\vec{r})} = \frac{2}{3} v^* \hbar \sqrt{\pi}\, \frac{3}{2}\sqrt{n(\vec{r})} + v(\vec{r}) = \mu$

2p

$$\Rightarrow n(\vec{r}) = \frac{1}{\pi}\left[\frac{\mu - v(\vec{r})}{\hbar v^*}\right]^2 , \quad \forall r : v(r) < \mu$$

$$\boxed{n(\vec{r}) = \frac{1}{\pi \hbar^2 v^{*2}}\left[\mu - v(r)\right]^2}$$

⑤ $\chi_0(\bar{r},\bar{r}') = \left.\frac{\delta n(\bar{r})}{\delta \upsilon(\bar{r}')}\right|_{\upsilon=0} = \frac{\delta}{\delta \upsilon(\bar{r}')}\left[\frac{1}{\pi \hbar^2 \upsilon^{*2}}\left(\mu - \upsilon(\bar{r})\right)^2\right]$

3P

$$= \frac{-1}{\pi(\hbar \upsilon^*)^2}\, 2\left(\mu - \upsilon(r)\right)\Big|_{\upsilon=0} \delta(\bar{r}-\bar{r}') = -\frac{2\mu}{\pi(\hbar \upsilon^*)^2}\, \delta(\bar{r}-\bar{r}')$$

$$\boxed{\chi_0(\bar{r},\bar{r}') = -\frac{2\mu}{\pi(\hbar \upsilon^*)^2}\, \delta(\bar{r}-\bar{r}') = \chi_0(\bar{r}-\bar{r}')}$$

⑥ $\chi(\bar{q}) = \int d\bar{r}\, \chi(r) e^{i\bar{q}\cdot\bar{r}} = -\int d\bar{r}\, \frac{2\mu}{\pi(\hbar \upsilon^*)^2}\, \delta(\bar{r}) e^{i\bar{q}\cdot\bar{r}} = -\frac{2\mu}{\pi(\hbar \upsilon^*)^2}$

2P

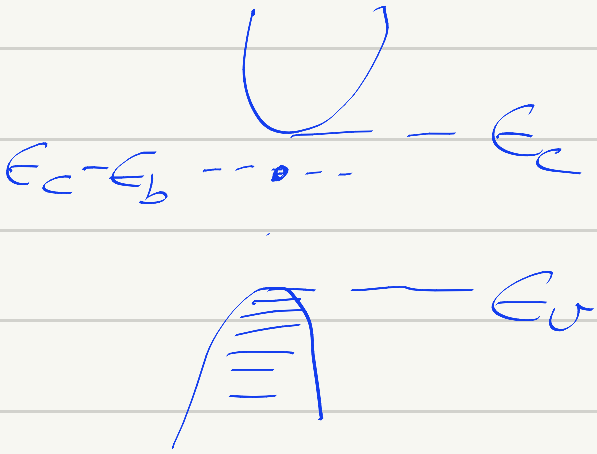
$$\boxed{\chi(\bar{q}) = -\frac{2\mu}{\pi(\hbar \upsilon^*)^2}}$$

ESERCIZIO 2

① We occupy the levels lowest in energy. Therefore

2p

$p_v = 0$, $n_d = N_d$, $n_c = 0$



② $T \to 0$ $\quad n_d(T) = \dfrac{N_d}{\frac{1}{2}e^{\beta(\epsilon_d - \mu)} + 1} = N_d$

3p

$$\Rightarrow \lim_{T\to 0} \frac{1}{2}e^{\beta(\epsilon_d-\mu)} = 0 \quad \Rightarrow \lim_{T\to 0} \beta(\epsilon_d-\mu) = -\infty$$

$$\Rightarrow \beta(\epsilon_d - \mu) \sim -\frac{\alpha}{T^\gamma} \quad , \quad \alpha, \gamma > 0$$

$$\Rightarrow \mu - \epsilon_d \sim \frac{\alpha k_B T}{T^\gamma} \quad \Rightarrow \mu = \epsilon_d + \alpha k_B T^{(1-\gamma)}$$

$$\boxed{T \to 0 : \mu \simeq \epsilon_d + \alpha k_B T^{1-\gamma} , \quad \alpha, \gamma > 0}$$

③ $n_c(T) = N_c(T) e^{-\beta(\epsilon_c - \mu)} = N_c(T) e^{-\beta\epsilon_b}$, $N_c(T) = \dfrac{1}{4}\left(\dfrac{2m_c k_B T}{\pi\hbar^2}\right)^{3/2}$

2p

④ $n_c(T^*) = N_d = N_c(T^*) e^{-\beta(\epsilon_c - \epsilon_d - \alpha k_B T^{*\,1-\gamma})} \simeq N_c(T^*) e^{-\beta^*\epsilon_b} = N_c(T^*) e^{-\frac{\epsilon_b}{k_B T^*}}$

3p

$$\Rightarrow \frac{\epsilon_b}{k_B T^*} = -\ln\left(\frac{N_d}{N_c(T^*)}\right) \Rightarrow T^* = \frac{\epsilon_b}{k_B \ln\left(\frac{N_c(T^*)}{N_d}\right)}$$

$$\boxed{T^* \simeq \frac{1}{k_B} \frac{\epsilon_b}{\ln\left(\frac{N_c(T^*)}{N_d}\right)}}$$

⑤ 2p

$$\frac{p_\sigma}{n_c} = \frac{P_\sigma e^{-\beta(\mu-\epsilon_\sigma)}}{N_c e^{-\beta(\epsilon_c-\mu)}} = \left(\frac{m_\sigma}{m_c}\right)^{3/2} \frac{e^{-\beta(\epsilon_c-\epsilon_b-\epsilon_\sigma)}}{e^{-\beta\epsilon_b}} = e^{-\beta(\epsilon_g-2\epsilon_b)}$$

$$\boxed{\frac{p_\sigma}{n_c} = e^{-\beta(\epsilon_g-2\epsilon_b)} \simeq e^{-\beta\epsilon_g}}$$

⑥ 3p $\quad n_c = N_d + p_\sigma \quad \Rightarrow \quad n_c - p_\sigma = N_d$

use

$$n_c = N_c e^{-\beta(\epsilon_c-\mu)} = N_c e^{-\beta(\epsilon_c-\mu_i)} e^{\beta(\mu-\mu_i)} = n_i e^{\beta(\mu-\mu_i)}$$

$$p_\sigma = P_\sigma e^{-\beta(\mu-\epsilon_\sigma)} = P_\sigma e^{-\beta(\mu_i-\epsilon_\sigma)} e^{-\beta(\mu-\mu_i)} = n_i e^{-\beta(\mu-\mu_i)}$$

to get

$$n_c - p_\sigma = 2n_i \sinh[\beta(\mu-\mu_i)] = N_d$$

$$\Rightarrow \beta(\mu-\mu_i) = \operatorname{arcsinh}\left(\frac{N_d}{2n_i}\right) \Rightarrow \mu = \mu_i + k_B T \operatorname{arcsinh}\left(\frac{N_d}{2n_i}\right)$$

- $\operatorname{arcsinh}(z) = \ln\left[z + \sqrt{1+z^2}\right]$

$$\Rightarrow \boxed{\mu = \mu_i + k_B T \ln\left[\frac{N_d}{2n_i} + \sqrt{1+\left(\frac{N_d}{2n_i}\right)^2}\right]}$$

For extrinsic regime, $N_d/n_i \gg 1$

$$\boxed{\mu = \mu_i + k_B T \ln \frac{N_d}{n_i}}$$