ESERCIZIO 1

① $E_I[\rho] = \int d\bar{r}\, \rho(\bar{r})\, \frac{2c}{3} \sqrt{\rho(\bar{r})}$

② $E[\rho] = \frac{2c}{3} \int d\bar{r}\, \rho(\bar{r})^{3/2} + \int d\bar{r}\, v(\bar{r})\, \rho(\bar{r})$

$$F[\rho] = E[\rho] - \int d\bar{r}\, \mu\, \rho(\bar{r})$$

$$0 = \frac{\delta F}{\delta \rho(\bar{r})} = \frac{\delta E}{\delta \rho(\bar{r})} - \mu$$

$$= c\sqrt{\rho(\bar{r})} + v(\bar{r}) - \mu$$

③ $\boxed{\sqrt{\rho(\bar{r})} = \frac{\mu - v(\bar{r})}{c}}$ ≥ 0 (a)

④ $(\rho + \delta\rho)^{3/2} = \rho^{3/2}\left[1 + \frac{\delta\rho}{\rho}\right]^{3/2}$

$$\simeq \rho^{3/2}\left[1 + \frac{3}{2}\frac{\delta\rho}{\rho} + \frac{3}{2}\left(\frac{1}{2}\right)\frac{1}{2}\left(\frac{\delta\rho}{\rho}\right)^2\right]$$

$$\Delta_2 F = \Delta_2 E_I = \frac{2c}{3}\int d\bar{r}\, \frac{3}{8}\, \frac{(\delta\rho(\bar{r}))^2}{\sqrt{\rho(\bar{r})}}$$

$$\Delta_2 F > 0 \quad \forall\, \delta\rho(\bar{r})$$

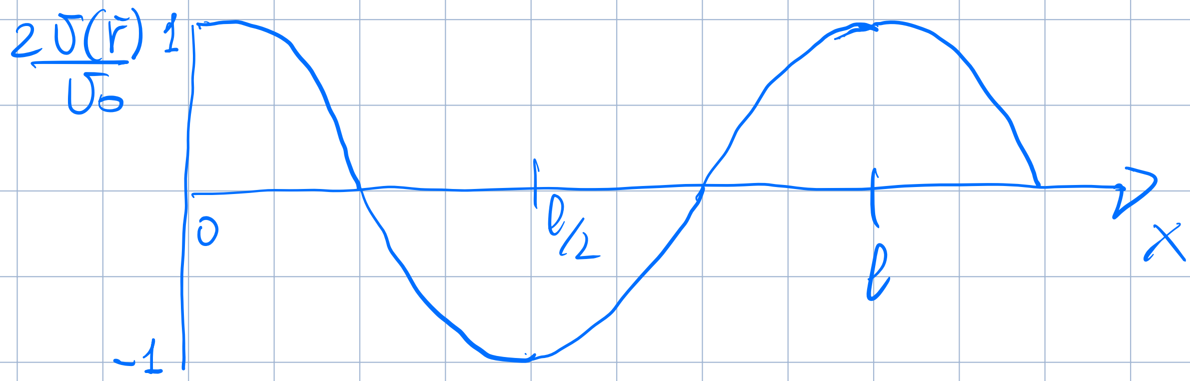
$$\Rightarrow \frac{c}{\sqrt{\rho(\bar{r})}} = \frac{c^2}{\mu - v(\bar{r})} > 0 \Rightarrow \boxed{\begin{array}{l} \mu - v(\bar{r}) \geq 0 \\ + (a) \Rightarrow \quad c \geq 0 \end{array}}\ (b)$$

⑤ $$\mu - \sigma(r) = \frac{\sigma_0}{2}\left[1 - \cos\left(\frac{2\pi x}{\ell}\right)\right] = \sigma_0 \sin^2\left(\frac{\pi x}{\ell}\right)$$

$$\rho(\bar{r}) = \left(\frac{\mu - \sigma(\bar{r})}{c}\right)^2 = \left(\frac{\sigma_0}{c}\right)^2 \sin^4\left(\frac{\pi x}{\ell}\right)$$

$$\boxed{\rho(\bar{r}) = \rho_0 \sin^4\left(\frac{\pi x}{\ell}\right), \quad \rho_0 = \left(\frac{\sigma_0}{c}\right)^2}$$

⑥



ESERCIZIO 2

① Fixed β_3 implies

$\beta_2^2 = 1-\beta_1^2-\beta_3^2$, function of β_1 for given β_3

thus

$$m(\bar{\beta}) = \sqrt{\frac{m_1 m_2 m_3}{m_1\beta_1^2 + (1-\beta_1^2-\beta_3^2)m_2 + m_3\beta_3^2}}$$

$$= \sqrt{\frac{m_1 m_2 m_3}{(m_1-m_2)\beta_1^2 + (1-\beta_3^2)m_2 + m_3\beta_3^2}}$$

Indipendence on β_1 implies $m_1-m_2=0$
or

$$\boxed{m_1 = m_2 \equiv m_{T'}}$$

Similarly varying β_2 at fixed β_3 implies

$$\beta_1^2 = 1-\beta_2^2-\beta_3^2$$

$$m(\bar{\beta}) = \sqrt{\frac{m_1 m_2 m_3}{m_1(1-\beta_2^2-\beta_3^2) + m_2\beta_2^2 + m_3\beta_3^2}}$$

$$= \sqrt{\frac{m_1 m_2 m_3}{m_1(1-\beta_3^2) + (m_2 - m_1)\beta_2^2 + m_3 \beta_3^2}}$$

The indipendence on β_2 implies $m_1 = m_2$!

(2) $\beta_3 = 1 \Rightarrow \beta_1 = \beta_2 = 0$

$$m^* = \sqrt{\frac{m_1 m_2 m_3}{m_3}} = \sqrt{m_1 m_2}$$

$$= \sqrt{m_\pi^2} = m_\pi$$

$$\Rightarrow m_\pi = 0.190\, m_e$$

(3) $\beta_3 = 0 \Rightarrow \beta_2^2 + \beta_1^2 = 1$

$$m^* = \sqrt{\frac{m_1 m_2 m_3}{m_1 \beta_1^2 + m_2 \beta_2^2}} = \sqrt{\frac{m_\pi^2 m_3}{m_\pi(\beta_1^2 + \beta_2^2)}}$$

$$= \sqrt{m_\pi m_3} = 0.418\, m_e$$

$$\Rightarrow m_3 = \frac{(0.418)^2 m_e^2}{m_\pi} \simeq \frac{(0.418)^2}{0.190} m_e$$

$$\boxed{m_3 \equiv m_L \simeq 0.920\, m_e}$$

(4) $$C = \frac{\sqrt{2}}{\hbar^3 \pi^2} \sqrt{m_{\pi}^2 m_L} = \frac{\sqrt{2}}{\hbar^3 \pi^2} \sqrt{\frac{m_L m_{\pi}^2}{m_e^3}}\, m_e^{3/2}$$

$$C = \frac{\sqrt{2}}{(1.05 \cdot 10^{-27})^3 \pi^2} \sqrt{0.920 \cdot (0.190)^2}\, (9.11 \cdot 10^{-28})^{3/2}$$

$$= 1.24 \cdot 10^{80} \times 0.182 \times 2.75 \cdot 10^{-41} = 6.21 \cdot 10^{38}$$

$$\boxed{C = 6.21 \cdot 10^{38}\ cm^{-3}\, erg^{-3/2}}$$

(5) With reference to the lesson on orbits of carrier at band extrema in a magnetico field (see material on the course webpage) we have

$$v_i = \gamma_i e^{\alpha t} \quad ; \qquad \alpha = \pm i \omega_c^* ,\ 0 \quad ;$$

$$\omega_c^* = \frac{eH}{m^* c} \quad ; \quad \delta = \pm \frac{i}{m^*} \quad .$$

since $\vec{\beta} = (0,1,0)$, $m^* = \sqrt{\frac{m_L m_{\pi}^2}{m_{\pi}}} = \sqrt{m_L m_{\pi}}$

we get

$$\begin{pmatrix} m_1\delta & 0 & 1 \\ 0 & m_2\delta & 0 \\ -1 & 0 & m_3\delta \end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\delta = 0 \Rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix}\gamma_3\\ 0\\ -\gamma_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow \quad \gamma_2 = (0, 1, 0) \Rightarrow \sigma_y = \text{costante}$$

$$\delta = \pm\frac{i}{m^*} \Rightarrow \begin{pmatrix} \pm i\frac{m_{T1}}{m^*} & 0 & 1 \\ 0 & \pm i\frac{m_{T1}}{m^*} & 0 \\ -1 & 0 & \pm i\frac{m_L}{m^*} \end{pmatrix} =$$

$$= \begin{pmatrix} \pm i\sqrt{\frac{m_{T1}}{m_L}} & 0 & 1 \\ 0 & \pm i\sqrt{\frac{m_{T1}}{m_L}} & 0 \\ -1 & 0 & \pm i\sqrt{\frac{m_L}{m_{T1}}} \end{pmatrix} = A$$

$$\Rightarrow \quad A\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix} \pm i\sqrt{\frac{m_{T1}}{m_L}}\gamma_1 + \gamma_3 \\ \pm i\sqrt{\frac{m_{T1}}{m_L}}\gamma_2 \\ -\gamma_1 \pm i\sqrt{\frac{m_L}{m_{T1}}}\gamma_3 \end{pmatrix}$$

$$\Rightarrow \quad \gamma_2 = 0 \quad ; \quad \gamma_3 = \mp i \sqrt{\frac{m_\pi}{m_L}}\, \gamma_1$$

$$\bar{\gamma} = \gamma_1 \left(1, \; 0, \; \mp \sqrt{\frac{m_\pi}{m_L}}\, i \right)$$

$$\bar{v}(t) = \mathrm{Re}\left\{ \gamma_1 \left(1, 0, \mp \sqrt{\frac{m_\pi}{m_L}}\, e^{i\frac{\pi}{2}} \right) e^{\pm i \omega_c^* t} \right\}$$

With γ_1 real we get

$$\bar{v}(t) = \gamma_1 \left(\cos \omega_c^* t, 0, \mp \sqrt{\frac{m_\pi}{m_L}} \cos\left(\omega_c^* t \pm \frac{\pi}{2} \right) \right)$$

$$\mp \cos\left(\omega_c^* t \pm \frac{\pi}{2} \right) = \sin \omega_c^* t$$

$$\bar{v}(t) = \gamma_1 \left(\cos \omega_c^* t, 0, \sqrt{\frac{m_\pi}{m_L}} \sin \omega_c^* t \right)$$

$$\bar{v}(t) = v_0 \left(\cos \omega_c^* t, 0, \sqrt{\frac{m_\pi}{m_L}} \sin \omega_c^* t \right)$$

$$\Delta \bar{r} = \bar{r}(t) - \bar{r}(0) = \frac{v_0}{\omega_c^*} \left(\sin \omega_c^* t, 0, \sqrt{\frac{m_L}{m_\pi}} \cos \omega_c^* t \right)$$

$$\Delta \bar{r}(t) \quad (\Delta x(t), 0, \Delta z(t))$$

$$\Rightarrow \quad (\Delta x(t))^2 + \frac{m_\pi}{m_L} (\Delta y(t))^2 = \left(\frac{v_0}{\omega_c^*} \right)^2$$

$$\Rightarrow \quad \frac{(\Delta x(t))^2}{m_\pi} + \frac{(\Delta y(t))^2}{m_L} = \frac{v_0^2}{m_\pi \omega_c^{*2}}$$

$$\Rightarrow \quad \boxed{\frac{(\Delta x(t))^2}{a^2} + \frac{(\Delta y(t))^2}{b^2} = 1}$$

$$a^2 = \frac{v_0^2}{\omega_c^{*2}} \, , \quad b^2 = \frac{v_0^2}{\frac{m_L}{m_\pi} \omega_c^{*2}}$$

The orbit is an "elliptical" spiral.

(6) $$m^* = \sqrt{\frac{m_\pi^2 m_L}{\frac{1}{3}(2m_\pi + m_L)}} = \sqrt{\frac{3(0.19)^2 0.92}{2 \cdot 0.19 + 0.92}} \, m_e$$

$$= \sqrt{\frac{0.0996}{1.30}} \, m_e = 0.277 \, m_e$$

$$\omega_c^* = \frac{eH}{m^* c} = \frac{4.80 \cdot 10^{-10} \cdot 1}{0.277 \cdot 9.11 \cdot 10^{-28} \cdot 3 \cdot 10^{10}}$$

$$= \frac{4.80 \cdot 10^{-10}}{7.57 \cdot 10^{-18}} = 6.34 \cdot 10^{7} \, s^{-1}$$

$$k_c^* = \frac{\omega_c^*}{c} = 2.11 \cdot 10^{-3} \, cm^{-1}$$