ESERCIZIO 1

1. $E(S_z) = E_0 + \mu_B g S_z H$

$$Z = \sum_{S_z=-S}^{S} e^{-\beta(E_0 + \mu_B g S_z H)} \qquad \boxed{S_z = S_z' - S}$$

$$Z = e^{-\beta E_0} e^{\beta \mu_B g S H} \sum_{S_z'=0}^{2S} e^{-\beta \mu_B g H S_z'}$$

$$= e^{-\beta E_0} e^{\beta \mu_B g H S} \frac{1 - e^{-\beta \mu_B g H (2S+1)}}{1 - e^{-\beta \mu_B g H}}$$

$$= e^{-\beta E_0} \frac{e^{\beta \mu_B g H (S+\frac{1}{2})} - e^{-\beta \mu_B g H (S+\frac{1}{2})}}{e^{\beta \mu_B g H \cdot \frac{1}{2}} - e^{-\beta \mu_B g H \frac{1}{2}}}$$

$$Z = e^{-\beta E_0} \frac{\sinh\left(\beta \mu_B g H S \left[1 + \frac{1}{2S}\right]\right)}{\sinh\left(\beta \mu_B g H S \cdot \frac{1}{2S}\right)}$$

$$\boxed{F = -k_B T \ln Z = E_0 - k_B T \ln \left[\frac{\sinh\left(x\left[1+\frac{1}{2S}\right]\right)}{\sinh\left(x \cdot \frac{1}{2S}\right)}\right]}$$

$x = \beta \mu_B g S H$

$$A(x) = \ln \left[\frac{\sinh\left[x\left(1+\frac{1}{2S}\right)\right]}{\sinh\left[x \cdot \frac{1}{2S}\right]}\right]$$

② $$M(H) = -\left.\frac{\partial F}{\partial H}\right|_{T'} = -\left.\frac{\partial x}{\partial H}\right|_{T'}\frac{\partial F}{\partial x} = -\frac{x}{H}(-k_B T')A'(x)$$

$$= -\frac{x}{H}(-k_B T')\left[\left(1+\frac{1}{2S}\right)\coth\left[x\left(1+\frac{1}{2S}\right)\right] - \frac{1}{2S}\coth\left[x\frac{1}{2S}\right]\right]$$

$$M(x) = \frac{x}{\beta H}A'(x) = \frac{x}{\beta H}B_S(x)$$

$$B_S(x) = A'(x) = \left(1+\frac{1}{2S}\right)\coth\left[x\left(1+\frac{1}{2S}\right)\right] - \frac{1}{2S}\coth\left[x\cdot\frac{1}{2S}\right]$$

$$\boxed{M(H) = \mu_B g S\, B_S(x)}$$

③ $$\chi(H) = \left.\frac{\partial M(H)}{\partial H}\right|_{T'} = \mu_B g S\left.\frac{\partial x}{\partial H}\right|_{T'}B_S'(x)$$

$$\boxed{\chi(H) = \beta(\mu_B g S)^2 B_S'(x)} = \beta(\mu_B g S)^2 A''(x)$$

④ $$S = -\left.\frac{\partial F}{\partial T'}\right|_{H} = k_B A(x) + k_B T'\frac{\partial x}{\partial T'}A'(x)$$

$$= k_B A(x) + k_B T'\left(-\frac{x}{T'}\right)A'(x)$$

$$\boxed{S = k_B\left[A(x) - x A'(x)\right]}$$

(5) $$C_H = T' \frac{\partial S}{\partial T'}\Big|_H = k_B T' \frac{\partial X}{\partial T'} \frac{\partial}{\partial X}\left[A(X) - X A'(X)\right]$$

$$= k_B T' \left(-\frac{X}{T'}\right)\left[\cancel{A'(X)} - \cancel{A'(X)} - X A''(X)\right]$$

$$\boxed{C_H = k_B X^2 A''(X) = k_B X^2 B_S'(X)}$$

$$\chi(H) = \beta(\mu_B g S)^2 B_S'(X) = \frac{X^2}{\beta H^2} B_S'(X)$$

$$= k_B T' \frac{X^2}{H^2} B_S'(X) = \frac{T'}{H^2} k_B X^2 B_S'(X)$$

$$= \frac{T'}{H^2} C_H$$

$$\boxed{C_H = \frac{H^2}{T'} \chi(H)}$$

(6) $$\left(\frac{g \mu_B H}{k_B \Theta_D}\right)^{2/5} = \left(\frac{2 \cdot 0{,}579 \cdot 10^{-8} \cdot 10^{4}}{8.62 \cdot 10^{-5} \cdot 200}\right)^{2/5}$$

$$= \left(\frac{0{,}0672 \cdot 10^{-4}}{10^{-3}}\right)^{2/5} = \left(6.72 \cdot 10^{-3}\right)^{2/5}$$

$$= 0.135 \qquad \boxed{\left(\frac{g \mu_B H}{k_B \Theta_D}\right)^{2/5} = 0.135}$$

CONTO NON RICHIESTO

$$x \ll 1 \Rightarrow B_S(x) \simeq \frac{1}{3}\frac{S+1}{S}x$$

$$C_H = k_B x^2 B_S'(x) \simeq k_B x^2 \frac{S+1}{3S}$$

$$C_H \simeq \frac{1}{3}k_B \left(\frac{g\mu_B H}{k_B T'}\right)^2 S(S+1)$$

$$C_H' = \frac{N_p}{V} C_H = \frac{1}{3}\frac{N_p}{V} k_B \left(\frac{g\mu_B H}{k_B T'}\right)^2 S(S+1)$$

$$C_V'(T') = \frac{N}{V}\, 234 \left(\frac{T'}{\Theta_D}\right)^3 k_B$$

$$C_V(T_0) = C_H \Longrightarrow$$

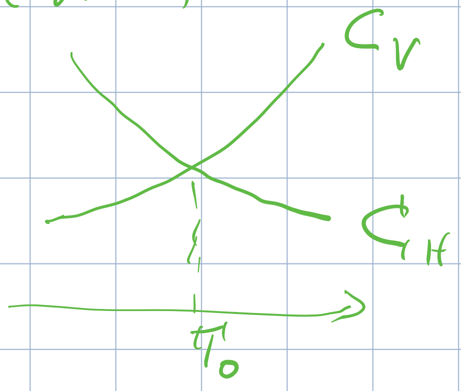
$$N\left(\frac{T_0}{\Theta_D}\right)^3 234 = \frac{N_p}{3}\left(\frac{g\mu_B H}{k_B T_0}\right)^2 S(S+1)$$

$$T_0^5 = \frac{N_p}{N}\frac{S(S+1)}{702}\left(\frac{g\mu_B H}{k_B \Theta_D}\right)^2 \Theta_D^5$$

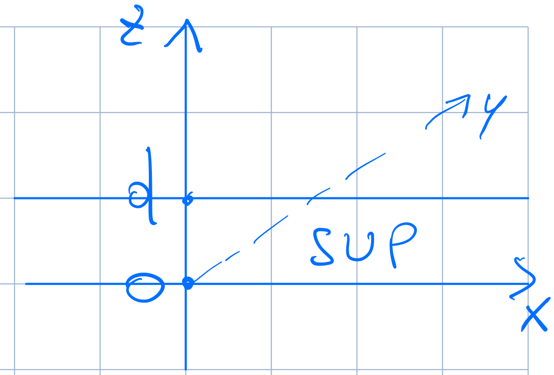
$$\boxed{T_0 = \left[\frac{N_p}{N}(S+1)S\right]^{\frac{1}{5}} \frac{1}{3.71}\left(\frac{g\mu_B H}{k_B \Theta_D}\right)^{2/5} \cdot \Theta_D}$$

A $T' < T_0$ domina C_H', in quanto

c_H è decrescente in T' e c_V è crescente in T'

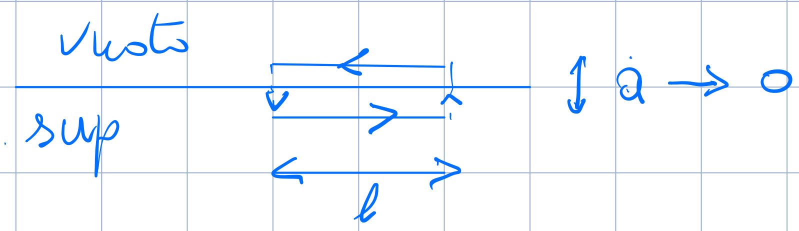


ESERCIZIO 2



Il superconduttore è delimitato dai piani
$z=0$ e $z=d$

① t)



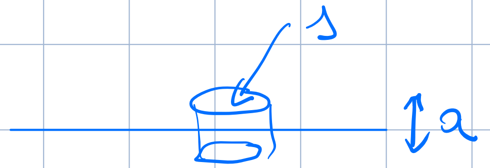
$$\int d\vec{l}\cdot\vec{B}(l) \simeq \left(B^t_{in} - B^t_{out}\right) l =$$

$$= \int d\vec{S}\cdot\nabla\times\vec{B} \propto \int d\vec{S}\,\vec{j}(\vec{S}) \xrightarrow[a\to 0]{} 0$$

si esclude una corrente a δ di Dirac nel piano di separazione

$$B^t_{in} = B^t_{out}$$

n)



$$\int d\vec{S}\cdot\vec{B} \simeq \left(B^n_{in} - B^n_{out}\right) s =$$

$$\int dV\,\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad B^n_{in} = B^n_{out}$$

B^n componente normale, B^t tangenziale

②

all'interno del superconduttore

$$\nabla^2 \vec{B} = \frac{\vec{B}}{\Lambda^2} \Rightarrow \nabla^2 B_\alpha = \frac{B_\alpha}{\lambda^2}, \quad \alpha = x, y, z$$

Evidentemente l'equazione

$$\nabla^2 B_\alpha = \frac{B_\alpha}{\lambda^2}$$

è separabile. Quindi

$$B_\alpha = a_1(x)\, a_2(y)\, a_3(z)$$

$$\frac{a_i''}{a_i} = \frac{1}{\lambda_i^2}, \quad \sum_{i=1}^{3} \frac{1}{\lambda_i^2} = \frac{1}{\Lambda^2} \qquad (1)$$

Ne consegue

$$B_\alpha(\vec{r}) = e^{x/\lambda_1} \cdot e^{y/\lambda_2} \cdot e^{z/\lambda_3} \qquad (2)$$

Il fatto che B_α è reale implica $1/\lambda_i$ reale e quindi $1/\lambda_i^2 \geq 0$.

Il fatto che $x, y \in R$ implica $1/\lambda_1 = 1/\lambda_2 = 0$ per evitare campi B_α divergenti a $\pm\infty$.

Infine otteniamo soluzioni accettabili e indipendenti

$$B_\alpha = e^{\pm z/\lambda}$$

Infine la soluzione generale a

$$\nabla^2 B_\alpha = \frac{1}{\lambda^2} B_\alpha \quad \text{è}$$

a) $B_\alpha(z) = A_+ e^{z/\lambda} + A_- e^{-z/\lambda}$ oppure

b) $B_\alpha(z) = A_1 \cosh(z/\lambda) + A_2 \sinh(z/\lambda)$

Nel seguito useremo la b)

(3)

Le condizioni al contorno implicano

$$\begin{cases} B_x(0) = H_0 = A_1 \\ B_x(d) = H_0 = A_1 \cosh(d/\lambda) + A_2 \sinh\left(\frac{d}{\lambda}\right) \end{cases}$$

$$A_2 = H_0 \left(1 - \cosh\left(\frac{d}{\lambda}\right)\right) \frac{1}{\sinh\left(\frac{d}{\lambda}\right)}$$

$$B_x(z) = H_0 \cosh\left(\frac{z}{\lambda}\right) + H_0\left(1 - \cosh\left(\frac{d}{\lambda}\right)\right) \frac{\sinh\left(\frac{z}{\lambda}\right)}{\sinh\left(\frac{d}{\lambda}\right)}$$

$$= \frac{H_0}{\sinh(d/\lambda)} \left[\sinh\frac{d}{\lambda} \cosh\frac{z}{\lambda} - \cosh\frac{d}{\lambda} \sinh\frac{z}{\lambda} + \sinh\frac{z}{\lambda} \right]$$

Infine utilizzando

$$\sinh(\alpha+\beta) = \sinh(\alpha)\cosh(\beta) + \cosh(\alpha)\sinh(\beta)$$

otteniamo

$$B_x(z) = \frac{H_0}{\sinh\frac{d}{\lambda}}\left[\sinh\frac{d-z}{\lambda} + \sinh\frac{z}{\lambda}\right]$$

Notiamo che le condizioni di continuità danno immediatamente

$B_y = B_z = 0$

(4)

$$\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B}$$

$$\frac{4\pi}{c}J_x = \partial_y B_z - \partial_z B_y = 0$$

$$\frac{4\pi}{c}J_z = \partial_x B_y - \partial_y B_x = 0$$

$$\frac{4\pi}{c}J_y = \partial_z B_x - \partial_x B_z$$

$$= \frac{H_0/\lambda}{\sinh\frac{d}{\lambda}}\left[\cosh\frac{z}{\lambda} - \cosh\left(\frac{d-z}{\lambda}\right)\right]$$

$$\boxed{J_y = \frac{cH_0}{\lambda 4\pi} \frac{1}{\sinh\left(\frac{d}{\lambda}\right)} \left[\cosh\frac{z}{\lambda} - \cosh\left(\frac{d-z}{\lambda}\right)\right]}$$

5

$$M(z) = \frac{H_0}{4\pi} \left\{ \frac{\sinh\left(\frac{d-z}{\lambda}\right) + \sinh\left(\frac{z}{\lambda}\right) - \sinh\left(\frac{d}{\lambda}\right)}{\sinh\left(\frac{d}{\lambda}\right)} \right\}$$

$$M_{av}(d) = \frac{1}{d} \int_0^d M(z)\, dz$$

$$= \frac{H_0 \ \lambda}{4\pi d \sinh\left(\frac{d}{\lambda}\right)} \left\{ -\cosh\left(\frac{d-z}{\lambda}\right) + \cosh\left(\frac{z}{\lambda}\right) - z \sinh\left(\frac{d}{\lambda}\right) \right\}_0^d$$

$$= \frac{H_0 \ \lambda}{4\pi d \sinh\left(\frac{d}{\lambda}\right)} \left\{ -1 + \cosh\frac{d}{\lambda} - \frac{d}{\lambda}\sinh\frac{d}{\lambda} + \cosh\frac{d}{\lambda} - 1 \right\}$$

$$= \frac{H_0 \ \lambda}{4\pi d \sinh\left(\frac{d}{\lambda}\right)} \left\{ 2\left[\cosh\frac{d}{\lambda} - 1\right] - \frac{d}{\lambda}\sinh\frac{d}{\lambda} \right\}$$

$$= \frac{H_0 \ \lambda}{4\pi d \sinh\left(\frac{d}{\lambda}\right)} \left\{ 4\sinh^2\frac{d}{2\lambda} - \frac{d}{\lambda}\sinh\frac{d}{\lambda} \right\}$$

$$= \frac{H_0 \ \lambda}{4\pi d} \left\{ \frac{4\sinh^2(d/2\lambda)}{2\sinh\frac{d}{2\lambda}\cosh\frac{d}{2\lambda}} - \frac{d}{\lambda} \right\}$$

$$\boxed{M_{av}(d) = -\frac{H_0}{4\pi}\left\{1 - \frac{2\lambda}{d}\,\mathrm{tgh}\,\frac{d}{2\lambda}\right\}}$$

6 $\frac{d}{\lambda} \ll 1$

Ricordiamo $\quad \mathrm{tgh}\,x \simeq x\left(1 - x^2/3\right), \; x \ll 1$

$$M_{av}(d) \simeq -\frac{H_0}{4\pi}\left[1 - 1 + \left(\frac{d}{2\lambda}\right)^2 \cdot \frac{1}{3}\right]$$

$$M_{av}(d) \simeq -\frac{H_0}{48\pi}\left(\frac{d}{\lambda}\right)^2$$

$\frac{d}{\lambda} \gg 1$

$$M_{av}(d) \simeq -\frac{H_0}{4\pi}\left[1 - \frac{2\lambda}{d}\right]$$

Nota: abbiamo usato

$$2\sinh(\alpha)\cosh(\alpha) = \sinh(2\alpha)$$