ESERCIZIO 1

① CAMBIO DI VARIABILE SUGGERITO

$$H_2 = \frac{P^2}{2M} + \frac{p^2}{2\mu} + \frac{K}{2} r^2 . \qquad (a)$$

$M = 2m$, $\mu = m/2$, $\vec{P}$ impulso CM

$\vec{p}$ impulso del moto relativo

$$Q_2(V,T) = \frac{1}{h^6} \int d\bar{P}\, e^{-\beta P^2/4m} \int dp\, e^{-\beta p^2/m} \times$$

$$\times \int_V d\bar{R} \int d\bar{r}\, e^{-\beta K r^2/2} =$$

$$= \frac{1}{h^6} (4 m K_B T \pi)^{3/2} (m K_B T \pi)^{3/2} V \left(\frac{2 K_B T \pi}{K}\right)^{3/2}$$

$$= \left(\frac{2\pi m K_B T}{h^2}\right)^3 V \left(\frac{2\pi K_B T}{K}\right)^{3/2}$$

$$= \left(\frac{2\pi m K_B T}{h^2}\right)^{3/2} V \left(\frac{(2\pi K_B T)^2 m}{h^2 K}\right)^{3/2}$$

$$= \frac{1}{\lambda^3} V \left(\frac{2\pi K_B T}{h} \sqrt{\frac{m}{K}}\right)^3 = \frac{1}{\lambda^3} V \left(\frac{K_B T}{\hbar} \sqrt{\frac{m}{K}}\right)^3$$

$$\boxed{Q_2(V\,T) = \frac{V}{\lambda^3} \left(\frac{K_B T}{\hbar} \sqrt{\frac{m}{K}}\right)^3}$$

②

$$Q_N(V,T) = \frac{1}{N!}\left[Q_1(V,T)\right]^N = \frac{1}{N!}\left[\frac{V}{\lambda^3}\left(\frac{k_B T}{\hbar}\sqrt{\frac{m}{k}}\right)^3\right]^N$$

$$\boxed{A(N,V,T) = -k_B T N \ln\left[\frac{e}{N}\frac{V}{\lambda^3}\left(\frac{k_B T}{\hbar}\sqrt{\frac{m}{k}}\right)^3\right]}$$

③

$$S = -\left.\frac{\partial A}{\partial T}\right|_{N,V} = N k_B \left\{ \ln\left[\frac{eV}{N\lambda^3}\left(\frac{k_B T}{\hbar}\sqrt{\frac{m}{k}}\right)^3\right] + T\frac{\partial}{\partial T}\ln\left(T^{9/2}\right)\right\} =$$

$$= N k_B\left\{\ln\left[\frac{eV}{N\lambda^3}\left(\frac{k_B T}{\hbar}\sqrt{\frac{m}{k}}\right)^3\right] + \frac{9}{2}\right\}$$

$dE = T\,dS$ a V, N costanti

$$\frac{1}{V}\left.\frac{dE}{dT}\right|_{N,V} = \frac{T}{V}\left.\frac{\partial S}{\partial T}\right|_{N,V} = \frac{N k_B T}{V}\left[\frac{\partial}{\partial T}\ln\left(T^{9/2}\right)\right]$$

$$\boxed{c_V = \frac{9}{2}\frac{N k_B}{V}}$$

Il modo più diretto per ottenere c_V è contare i termini quadratici nell'equazione (a): sono 9 termini!

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$$\langle \bar{r}^2 \rangle = \frac{\int d\bar{r}\, e^{-\frac{\beta\kappa r^2}{2}} r^2}{\int d\bar{r}\, e^{-\frac{\beta\kappa r^2}{2}}}$$

$$= \frac{\cancel{4\pi} \int_0^\infty dr\, r^2 \cdot e^{-\frac{\beta\kappa}{2} r^2} r^2}{\cancel{4\pi} \int_0^\infty dr\, r^2\, e^{-\frac{\beta\kappa}{2} r^2}}$$

$$= \left.\frac{\frac{\partial^2}{\partial\alpha^2} \int_0^\infty dr\, e^{-\alpha r^2}}{-\frac{\partial}{\partial\alpha} \int_0^\infty dr\, e^{-\alpha r^2}}\right|_{\alpha = \frac{\beta\kappa}{2}} = \frac{\frac{\partial}{\partial\alpha^2} \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}}{-\frac{\partial}{\partial\alpha} \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}} = \frac{\frac{\partial}{\partial\alpha^2} \alpha^{-\frac{1}{2}}}{-\frac{\partial}{\partial\alpha} \alpha^{-\frac{1}{2}}}$$

$$= \frac{\frac{3}{4}\alpha^{-5/2}}{\frac{1}{2}\alpha^{-3/2}} = \frac{3}{2}\frac{1}{\alpha} = \frac{3}{2}\frac{2}{\beta\kappa} = \frac{3k_B T}{\kappa}$$

$$\boxed{\langle r^2 \rangle = \frac{3k_B T}{\kappa}}$$

ESERCIZIO 2

① $\mathcal{H}_1 = \frac{p_x^2}{2m_x} + \frac{p_y^2}{2m_y} + \frac{K}{2}(x^2+y^2)$

$Q_N = \frac{1}{N!} q^N$

$$q = \frac{1}{h^2}\int_{-\infty}^{+\infty} dp_x \, e^{-\beta \frac{p_x^2}{2m_x}} \int_{-\infty}^{+\infty} dp_y \, e^{-\beta \frac{p_y^2}{2m}} \times$$

$$\times \int_{-\infty}^{+\infty} dx \, e^{-\frac{\beta K x^2}{2}} \int_{-\infty}^{\infty} dy \, e^{-\frac{\beta K y^2}{2}}$$

$$= \frac{1}{h^2}\sqrt{2\pi m_x K_B T} \sqrt{2\pi m_y K_B T}\left(\sqrt{\frac{2\pi K_B T}{K}}\right)^2$$

$$= \frac{1}{h^2} \frac{(2\pi K_B T)^2 \sqrt{m_x m_y}}{K} = (K_B T)^2 \frac{m^*}{K}\left(\frac{2\pi}{h}\right)^2$$

$$= \left(\frac{K_B T}{\hbar\omega}\right)^2$$

$$Q(N,T) = \frac{1}{N!}\left(\frac{K_B T}{\hbar\omega}\right)^{2N}$$

$$\boxed{Q(N,T) = \frac{1}{N!}\left[\left(\frac{K_B T}{\hbar\omega}\right)^2\right]^N \simeq \left[\frac{e}{N}\left(\frac{K_B T}{\hbar\omega}\right)^2\right]^N}$$

$$\boxed{A(N,T) = -N K_B T \ln\left[\frac{e}{N}\left(\frac{K_B T}{\hbar\omega}\right)^2\right]}$$

(2)

$$\lim_{N\to\infty} \frac{A(N,T)}{N} = -N k_B T \lim_{N\to\infty} \ln \frac{1}{N} = +\infty$$

Se A fosse estensiva

$$\lim_{N\to\infty} \frac{A(N,T)}{N} = f(T)$$ in genere finita

Ma usiamo la definizione di potenziale termodinamico estensivo.

In $A(N,T)$ l'unica variabile estensiva è N, quindi $N\partial A/\partial N$ deve essere uguale a A. Invece

$$N\frac{\partial A}{\partial N} = -N k_B T \left[\ln\left\{\frac{e}{N}\left(\frac{k_B T}{\hbar\omega}\right)^2\right\} + N\frac{\partial}{\partial N}\ln\frac{1}{N}\right]$$

$$= -N k_B T \left[\ln\left\{\frac{e}{N}\left(\frac{k_B T}{\hbar\omega}\right)^2\right\} - 1\right]$$

$$\neq A = -N k_B T \ln\left\{\frac{e}{N}\left(\frac{k_B T}{\hbar\omega}\right)^2\right\}.$$

(3)

$$\langle \delta(\bar r - \bar r_1)\rangle = \frac{\int d\bar r_1\, \delta(\bar r - \bar r_1)\, e^{-\beta\frac{K}{2}(x_1^2+y_1^2)}}{\int d\bar r_1\, e^{-\beta\frac{K}{2}(x_1^2+y_1^2)}}$$

$$= \frac{e^{-\beta\frac{K}{2}r^2}}{2\pi k_B T/K}$$

$r^2 = x^2 + y^2$

$$\rho(r) = \frac{N e^{-\beta \kappa r^2/2}}{2\pi k_B T/\kappa} = N \frac{e^{-\beta m^* \omega^2 r^2/2}}{\frac{2\pi k_B T}{m^* \omega^2}}$$

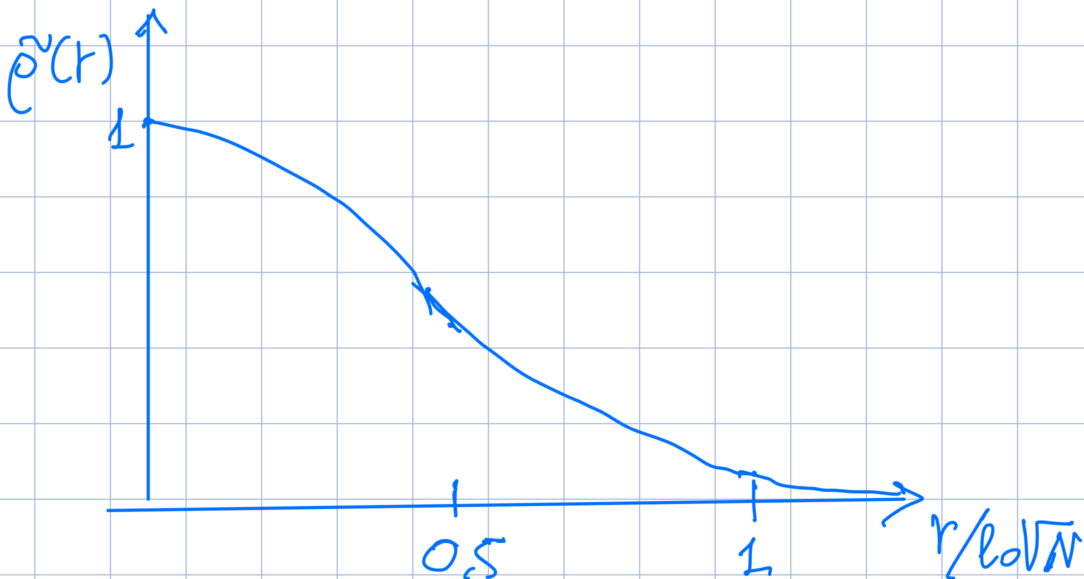
$$\boxed{\omega^2 \Rightarrow \frac{\omega_0^2}{N}}$$

$$\boxed{l_0^2 = \frac{2\pi k_B T}{m^* \omega_0^2}}$$

$$= \cancel{N} \frac{e^{-\beta m^* \omega_0^2 r^2/2N}}{2\pi k_B T} \frac{\omega_0^2 m^*}{\cancel{N}}$$

$$= \frac{1}{l_0^2} e^{-\frac{\pi}{l_0^2}\frac{r^2}{N}} = \frac{1}{l_0^2} e^{-\pi\left(\frac{r}{l_0\sqrt{N}}\right)^2}$$

$$\tilde{\rho}(r) = e^{-\pi\left(\frac{r}{l_0\sqrt{N}}\right)^2}$$



$$\tilde{\rho}(r^*) = \frac{1}{2} = e^{-\pi\left(r^*/l_0\sqrt{N}\right)^2}$$

$$-\pi\left(\frac{r^*}{l_0\sqrt{N}}\right)^2 = -\ln 2$$

$$r^* = \sqrt{\frac{\ln 2}{\pi}}\, l_0\sqrt{N} \sim \sqrt{N}$$

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$$S = -\frac{\partial A}{\partial T}\Big|_N = \frac{\partial}{\partial T}\left[N k_B T \ln \frac{e}{N}\left(\frac{k_B T}{\hbar\omega}\right)^2\right.$$

$$= N k_B \left[\ln \frac{e}{N}\left(\frac{k_B T}{\hbar\omega}\right)^2 + T\frac{\partial}{\partial T}\ln T^2\right]$$

$$= N k_B \left[\ln \left(\frac{e}{N}\frac{k_B T}{\hbar\omega}\right)^2 + 2\right]$$

$$S = N k_B \ln\left(\frac{e^3}{\cancel{N}}\left(\frac{k_B T}{\hbar\omega_0}\right)^2 \cancel{N}\right)$$

$$\frac{S}{N} = k_B \ln\left(e^3\left(\frac{k_B T}{\hbar\omega_0}\right)^2\right)$$

- S è estensiva
- In effetti è chiaro che

$$N\frac{\partial S}{\partial N} = S \qquad !$$