Some remarks on the use of the strict complementarity in checking coherence and extending coherent probabilities

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1. Introduction.

The problem of verifying the coherence of a probabilistic assignment on a finite set of conditional events has been recently examined by several authors ([6, 7, 10, 11, 25]). The algorithms proposed are substantially very similar and they all require to solve a sequence of systems of linear inequalities, such that the number of rows of each of these systems is determined by the solution found for the previous one. In the sequel, we shall examine how the concept of strictly complementary solution of a linear program (LP) can be applied in this framework to reduce the number and the size of the systems to be solved. We shall refer to the procedure proposed in [25], but the technique described can be employed in the other variants of the algorithm as well. Besides, we will see that the strict complementarity can be also applied, for the same purposes, to the algorithm for extending coherent probabilities proposed in [26].

The definition of strictly complementary solution of a LP dates back to the beginning of linear programming. Nevertheless, the importance of the strict complementarity and of the correlated concept of optimal partition have turned out only recently, especially as a tool for sensitivity analysis and parametric programming ([2, 17]). H.J. Greenberg ([14]) presents other examples of applications that draw advantage from an analysis performed by means of the strict complementarity.

The rediscovery of the strict complementarity is probably motivated by the fairly recent development of the interior point methods for solving linear programs, most of which converge to a strictly complementary solution. Indeed, these methods have received an increasing attention after the proposal of the algorithm of Karmarkar ([19]). For a long time the simplex has represented the principal and more used algorithm in the resolution of linear programming problems, owing to its practicalness and efficiency, though it is not a polynomial-time algorithm ([24]). On the contrary, most of the interior point methods have a polynomial resolution time and the good results they provide, in particular when large scale linear programs have to be faced, make them really competitive with the simplex algorithm. A unifying presentation of the principles underlying most of these methods and an analysis of their computational complexity is presented in [23, 27]. E.D. Andersen et al. ([1]) present an overview of the issues concerning the implementation of these methods and a comparison with the simplex.

The paper is organised as follows. In Section 2 the notation used in the sequel is introduced. Section 3 recalls some classical results of linear programming. Besides, a technique that determines a strictly complementary solution of a LP by means of a (basic) solution of a particular linear programming problem is presented. Section 4 illustrates some results that link the strict complementarity and the concept of implied equality in a system of linear inequalities. In Section 5 we show the use of the strict complementarity in implementing the algorithm for checking the coherence of a conditional probabilistic assessment proposed in [25]. Then, we also examine the application of the strict complementarity to the algorithm for extending a coherent probability described in [26]. We conclude with an example in Section 6.

2. Notations.

A^T will denote the transpose of a matrix A. Given a vector $x \in \Re^p$, $\sigma(x)$ will indicate the set of indexes of the positive components of x. With e we shall denote the vector of all one, whose dimension will be inferred by the context in which it is used.

Given $x, y \in \Re^p$ we set $x^T y = \sum_{i=1}^{p} x_i y_i$. Besides, $x \leq y$ ($x < y$) will indicate that $x_i \leq y_i$ ($x_i < y_i$) $\forall i = 1, \ldots, p$. Analogously, $x \geq 0$ ($x > 0$) means $x_i \geq 0$ ($x_i > 0$) $\forall i = 1, \ldots, p$.

If b is a vector of $\Re^q$ and A a real matrix of q rows and p columns, $Ax \leq b$ will denote a system of q linear inequalities. The generic i-th inequality of the system will be represented by $a_i x \leq b_i$. Similarly, we shall indicate with $Ax < b$ a system with only strict inequalities.

We will use a logical notation for operations and relations involving events. Symbols '$\wedge$','$\vee$' and '$\Rightarrow$' will be used to indicate the logical product, sum and implication of events respectively. Given an event E, $\overline{E}$ indicates its negation, while Ω and ϕ represent the sure event and the impossible one respectively. If E and F are events, we set $\delta(E \Rightarrow F)$ equal to one if it is $E \Rightarrow F$, to zero otherwise. $|E|$ denotes the random number equal to 1 if E is true, to zero otherwise.

Given a partition Π of Ω and an event $K \neq \phi$, we will denote with $\Pi|K$ the conditional partition formed by the atoms of Π conditioned on K, that is the atoms of $\Pi|K$ are formed by the atoms of Π still possible after conditioning on K. With $\mathcal{A}_L(\Pi)$ ($\mathcal{A}_L(\Pi|K)$) we will indicate the set of events logically dependent from Π ($\Pi|K$), that is the set of events which are logical sums of atoms of Π ($\Pi|K$). We also set $\mathcal{A}_L^{\phi}(\Pi) = \mathcal{A}_L(\Pi) - \{\phi\}$ and $\mathcal{A}_L(\Pi)|\mathcal{A}_L^{\phi}(\Pi) = \{E|H : E \in \mathcal{A}_L(\Pi), H \in \mathcal{A}_L^{\phi}(\Pi)\}$.

3. Optimal partition and strict complementarity.

Thms. 3.1, 3.2, 3.3 and Cor. 3.5 that follow summarise some known results of linear programming. We refer to [24] for a detailed exposition and proof of them.

Let A be a real matrix of q rows and p columns, $b \in \Re^q$, $c \in \Re^p$. Let $P = \{x \in \Re^p : Ax \leq b\}$ and $D = \{y \in \Re^q : A^T y = c, y \geq 0\}$. We indicate with (P) the problem $\max\{c^T x : x \in P\}$ and with (D) the dual problem $\min\{b^T y : y \in D\}$.

Theorem 3.1 (Duality theorem of linear programming). *(P) has an optimal solution iff (D) has an optimal one and, in this case, it is $\max\{c^T x : x \in P\} = \min\{b^T y : y \in D\}$. If (P) is infeasible, then (D) is either infeasible or unbounded. If (P) is unbounded, then (D) is infeasible.*

If $\overline{x}$ is an optimal solution for (P) and $\overline{y}$ is an optimal solution for (D), we will say that $(\overline{x}, \overline{y})$ is an *optimal pair* for (P) and (D). Given $x_0 \in P$ and $y_0 \in D$, the non-negative quantity $(b - Ax_0)^T y_0$ is called *duality gap.*

Theorem 3.2 (Complementary slackness). *Let $x_0 \in P$ and $y_0 \in D$. Then the following are equivalent*

(1) (x_0, y_0) *is an optimal pair for* (P) *and* (D)*;*

(2) $c^T x_0 = b^T y_0$*;*

(3) $(b - Ax_0)^T y_0 = 0$.

Thm. 3.2 states the vanishing property of the duality gap of an optimal pair, usually referred to as *complementary slackness.* If we set $I = \{1, ..., q\}$, $B = \{i \in I :$

$\exists\ x_0$ *optimal for* $(P) : a_i x_0 < b_i\}$ and $N = \{i \in I : \exists\ y_0$ *optimal for* $(D) : y_{0i} > 0\}$, it easily follows that it is $B \cap N = \emptyset$. Thm. 3.3 guarantees that it is also $B \cup N = I$.

Theorem 3.3 (Strict complementarity). *Let* (P) *and* (D) *be feasible. Then, for each inequality* $a_i x \le b_i$ *of the system* $Ax \le b$ *exactly one of the following holds*

(1) there exists an optimal solution of (P) x_0 *such that* $a_i x_0 < b_i$*;*

(2) there exists an optimal solution of (D) y_0 *such that* $y_{0i} > 0$.

Therefore, B and N form a partition of I, called *optimal partition* of the couple of primal-dual linear programs (P) and (D).

Definition 3.4. *An optimal pair* $(\overline{x}, \overline{y})$ *for* (P) *and* (D) *is said to be a pair of strictly complementary optimal solutions iff it is* $b - A\overline{x} + \overline{y} > 0$.

Let $\overline{x}$, $\overline{y}$ be feasible solutions of (P) and (D) respectively. Then, $(\overline{x}, \overline{y})$ is a pair of strictly complementary optimal solutions iff, for every $i = 1, \dots, q$, exactly one between $b_i - a_i\overline{x}$ and $\overline{y}_i$ is positive, while the other is null.

Consider for every $i \in B$ an optimal solution x^i of (P) such that $a_i x^i < b_i$ and for every $j \in N$ an optimal solution y^j of (D) such that $y^j_j > 0$. By considering convex combinations with positive coefficients of these x^i and y^j respectively, we obtain a strictly complementary optimal pair. Therefore, we have the following corollary of Thm. 3.3.

Corollary 3.5 (Existence of a strictly complementary optimal solution). *Let* (P) *and* (D) *be feasible. Then there exists a pair* $(\overline{x}, \overline{y})$ *of strictly complementary optimal solutions of* (P) *and* (D)*. Besides, for every other optimal pair* (x_0, y_0) *it is* $\sigma(b - Ax_0) \subseteq \sigma(b - A\overline{x})$ *and* $\sigma(y_0) \subseteq \sigma(\overline{y})$.

The problem of determining the optimal partition of a couple of primal-dual linear programs can be faced examining the optimal basic solutions, but this technique can require a remarkable computational effort ([13]). Observe that, if $(\overline{x}, \overline{y})$ is a pair of strictly complementary optimal solutions of (P) and (D), then $B = \sigma(b - A\overline{x})$ and $N = \sigma(\overline{y})$. Thus, the determining of a pair of strictly complementary optimal solutions or, clearly, of just one of them as well, allows to single out the optimal partition. Therefore, interior point methods for linear programming, most of which are characterised by the property of providing a strictly complementary optimal solution, are considered suitable to this purpose. A discussion about the identification of the optimal partition by means of several interior point algorithms can be found in [20]. See also [8] for an analysis of some computational aspects.

Observe that the simplex algorithm does not generally provide a strictly complementary optimal pair. In fact, the simplex does not supply it except if the optimal primal-dual solution is unique ([14]) and the uniqueness condition is seldom satisfied in practice. Notwithstanding this, a pair of strictly complementary optimal solutions can be determined also by resorting to the simplex algorithm. This possibility enables us to face the problem on the basis of the rich literature and of the well-tested techniques relative to this method. A way is presented in the sequel. At first, observe that the strictly complementary optimal solutions of (P) and (D) are the solutions of the system (Thm. 3.2 and Def. 3.4)

$$\begin{cases} Ax \leq b \\ A^T y = c \\ y \geq 0 \\ c^T x - b^T y = 0 \\ b - Ax + y > 0 \end{cases}$$

Prop. 3.6 generalises a technique to determine a strictly complementary optimal pair, employed by Freund in [9], that relies on the previous observation and on a well fit LP.

Proposition 3.6. *Consider the LP*

$$(LPSC) \qquad \begin{array}{l} \max\limits_{x,y,\alpha,u} \ u \\ Ax \leq b\alpha \\ A^T y = c\alpha \\ y \geq 0 \\ c^T x - b^T y = 0 \\ b\alpha - Ax + y \geq eu \\ u \leq 1 \\ \alpha \geq 1 \end{array}$$

If (P) and (D) are feasible, then $(LPSC)$ is feasible. If $(LPSC)$ is feasible then it is bounded and (P) and (D) are feasible. Besides, if $(x^*, y^*, \alpha^*, u^*)$ is an optimal solution of $(LPSC)$, then $(\frac{x^*}{\alpha^*}, \frac{y^*}{\alpha^*})$ is a pair of strictly complementary optimal solutions of (P) and (D).

Proof Let (P) and (D) be feasible. Then, by Cor. 3.5, there exists a pair of strictly complementary optimal solutions $(\overline{x}, \overline{y})$ for them. By defining $\delta = \min\{1, b_i - a_i\overline{x} + \overline{y}_i \ (i = 1, \dots, q)\}$, $(\frac{\overline{x}}{\delta}, \frac{\overline{y}}{\delta}, \frac{1}{\delta}, 1)$ is a feasible (and optimal) solution for $(LPSC)$ and the maximum objective value is 1. Suppose now $(LPSC)$ feasible. If $(x_0, y_0, \alpha_0, u_0)$ is a feasible solution for $(LPSC)$, then $\frac{x_0}{\alpha_0}$ and $\frac{y_0}{\alpha_0}$ are feasible (and optimal) solutions for (P) and (D) respectively and $(LPSC)$ is obviously bounded. Let $(x^*, y^*, \alpha^*, u^*)$ be an optimal solution for $(LPSC)$. Then, (P) and (D) are feasible and we have previously seen that it must be $u^* = 1$. Therefore, $b - A\frac{x^*}{\alpha^*} + \frac{y^*}{\alpha^*} \geq e\frac{1}{\alpha^*} > 0$. Thus, $(\frac{x^*}{\alpha^*}, \frac{y^*}{\alpha^*})$ is a pair of strictly complementary optimal solutions of (P) and (D). ▪

4. Some results about implied equalities.

The following definition introduces the concept of implied equality in a system of linear inequalities ([15]).

Definition 4.1. *A linear inequality $a_i x \leq b_i$ of a system $Ax \leq b$ is an implied equality iff the system is consistent and it is $a_i x^* = b_i$ for every solution x^* of $Ax \leq b$.*

Following the notation used in [24], with $A^= x \leq b^=$ we shall denote the system of implied equalities in $Ax \leq b$ and with $A^+ \leq b^+$ the system formed by the other inequalities. The next proposition follows easily.

Proposition 4.2. *If system $Ax \leq b$ is consistent, then it has a solution x^* such that $A^= x^* = b^=$ and $A^+ x^* < b^+$.*

Proof For every inequality $a_i x \leq b_i$ of the system $A^+ x \leq b^+$ there is a solution x^i such that $a_i x^i < b_i$. A convex combination with positive coefficients of these x^i is the required solution. ▪

Let A' be a real matrix of s rows and r columns and $b' \in \Re^s$. In Sect. 5 we will consider a system in the form $A'x = b', x \geq 0$. It can be written as $A'x \leq b', -A'x \leq b', -x' \leq 0$ and the first $2s$ inequalities are obviously implied equalities if it is consistent. In this hypothesis, by Prop. 4.2, there exists a solution x^* of $A'x = b', x \geq 0$ such that $x^*_j = 0$ iff $x_j \geq 0$ is an implied equality $(j = 1, \dots, r)$, that is $\{1, \dots, r\} - \sigma(x^*)$ is the set of indexes of the implied equalities among the $x_j \geq 0$ $(j = 1, \dots, r)$. Prop. 4.3 shows how to single out this set. It is sufficient to consider a strictly complementary solution of a LP whose feasible set is determined by the system itself and whose objective function is quite simple.

Proposition 4.3. *Consider the following couple of primal-dual linear programming problems*

$$\begin{array}{lll} \max \ b'^T y & & \min \ 0 \\ A'^T y \leq 0 \quad (LP1) & & A'x = b' \quad (DLP1) \\ & & x \geq 0 \end{array}$$

and suppose (DLP1) feasible. Let $\overline{x}$ be a strictly complementary optimal solution of $(DLP1)$. Then, $x_j \geq 0$ is an implied equality of $A'x = b', x \geq 0$ iff $j \notin \sigma(\overline{x})$ $(j = 1, \ldots, r)$.

Proof Observe that (LP1) is always feasible. Hence, by Cor. 3.5, there exists a pair of strictly complementary optimal solutions $(\overline{y}, \overline{x})$ for the problems and it is $\sigma(x_0) \subseteq \sigma(\overline{x})$ for every optimal solution x_0 of $(DLP1)$. Because of the choice of the objective function, the optimal solution set of $(DLP1)$ coincides with the set of solutions of the system $A'x = b', x \geq 0$. The result follows straightforward. ▪

An examination of the role of the strict complementarity in discovering the implied equalities of a system $Ax \leq b$ can be found in [9, 15]. Besides, [9] presents another technique for determining the implied equalities. It relies on finding a solution, not necessarily strictly complementary, of an appropriate LP. We report this result in a form adapted to examine a system $A'x = b', x \geq 0$ and refer to the cited report for the general result.

Proposition 4.4. *Given the system $A'x = b', x \geq 0$, consider the LP*

$$(LP2) \qquad \begin{array}{l} \max\limits_{x,y,u} \ e^T y \\ A'x = b'u \\ y \leq x \\ 0 \leq y \leq e \\ u \geq 1 \end{array}$$

Then

- *if the system $A'x = b', x \geq 0$ is consistent, (LP2) is feasible and bounded. Besides, given an optimal solution of $(LP2)$ $(\overline{x}, \overline{y}, \overline{u})$, $\frac{\overline{x}}{\overline{u}}$ is a solution of the system and $\forall j = 1, \ldots, r$, $x_j \geq 0$ is an implied equality iff $j \notin \sigma(\frac{\overline{x}}{\overline{u}})$;*
- *if the system $A'x = b', x' \geq 0$ is inconsistent, then $(LP2)$ is infeasible.*

5. Strict complementarity, checking coherence and extending coherent probabilities.

Before analysing the use of the strict complementarity in implementing an algorithm for checking the coherence of a probabilistic assignment on a finite set of conditional events, we recall the definition of coherent probability to which we will refer in the sequel ([16]).

Definition 5.1. *Let $\mathcal{F}$ be a set of conditional events. $P(\cdot|\cdot)$ is a coherent conditional probability on $\mathcal{F}$ iff $\forall m, \forall E_i|H_i \in \mathcal{F}, \forall s_i \in \Re, i = 1, \ldots, m$, defining $G = \sum_{i=1}^{m} s_i|H_i|(|E_i| - P(E_i|H_i))$ and $H = \bigvee_{i=1}^{m} H_i$, it is $\max G|H \geq 0$.*

Let $\mathcal{E} = \{E_1|H_1, \ldots, E_n|H_n\}$ be a finite set of conditional events and let $P(E_i|H_i) = p_i$ $(i = 1, \ldots, n)$ be an assignment of real numbers on $\mathcal{E}$. Let $\mathcal{P}$ be the partition formed by the not impossible logical products obtained developing the expression $\wedge_{i=1}^{n}[(E_i \wedge H_i) \vee (\overline{E}_i \wedge H_i) \vee \overline{H}_i]$. Define $I_1 = \{1, \ldots, n\}$, $K_1 = \vee_{i \in I_1} H_i$. If $\overline{K}_1 \neq \phi$, then $\overline{K}_1$ is an atom of $\mathcal{P}$. We will indicate with $e_1, e_2, \ldots, e_m$ the remaining atoms of the partition. We define also $J_1 = \{j : e_j \Rightarrow K_1\} = \{1, \ldots, m\}$.

The following algorithm ([25]) verifies the coherence of P.

1) Set h:=1.
2) Consider the following system

$$(S_h) \begin{cases} \displaystyle\sum_{j \in J_h} x_j \delta(e_j \Rightarrow E_i \wedge H_i) - p_i \sum_{j \in J_h} x_j \delta(e_j \Rightarrow H_i) = 0 \quad (i \in I_h) \\ \displaystyle\sum_{j \in J_h} x_j = 1, \ x_j \geq 0 \ (j \in J_h) \end{cases}$$

   If system (S_h) is inconsistent then the assigned probability is not coherent; stop.
3) Let $(x_j(h))_{j \in J_h}$ a solution of system (S_h).
4) Define
   $I_{h+1} = \{i \in I_h : \sum_{j \in J_h} x_j(h) \delta(e_j \Rightarrow H_i) = 0\}$,
   $K_{h+1} = \vee_{i \in I_{h+1}} H_i$,
   $J_{h+1} = \{j : e_j \Rightarrow K_{h+1}\}$.
5) If $I_{h+1} = \emptyset$ the assigned probability is coherent; stop.
6) Set h:=h+1.
7) Goto 2).

Observe that two major questions arise, when this algorithm is implemented. The first one regards the efficient determination of the partition $\mathcal{P}$. A procedure that determines it, taking account also of an assigned set of logical relations among the events to remove the impossible atoms, has been proposed in [3]. However, the number of atoms grows exponentially with the number of events considered, unless the set of known logical relations among these events enable us to exclude a consistent part of them during the procedure. Clearly, in the worst cases, it could be practically impossible to construct the matrix of the system (S_h), even with a relatively small number of events. In the analysis of strictly correlated problems, some authors ([18, 21]) have proposed to use a column generation algorithm that does not require the previous knowledge of the complete matrix of the problem to be solved.

The second question regards the technique to determine a solution of system (S_h). It is well known that a solution of a system of linear inequalities can be found with a polynomial-time algorithm. A proof of the polynomial equivalence between the resolution of a system of linear inequalities and linear programming is presented in [24]. Indeed, the usual technique employed is to maximise (or minimise) a suitable linear function with constraints formed by the system to be solved or to resort to a suitable auxiliary LP problem ([5]).

Nevertheless, by observing definitions of I_{h+1} and J_{h+1}, we see that the number of rows and columns of the system (S_{h+1}) depends on the solution found for (S_h). Let $\mathcal{S}_h$ be the set of solutions of (S_h). If we wish to reduce the number of rows of the system (S_{h+1}), we have to determine a solution $(x_j(h))_{j\in J_h}$ of (S_h) that reduces the number of events H_i such that $\sum_{j\in J_h} x_j(h)\delta(e_j \Rightarrow H_i) = 0$. A technique that determines I_{h+1} as the set $\{i \in I_h : \max \sum_{j\in J_h} x_j(h)\delta(e_j \Rightarrow H_i) = 0, (x_j(h))_{j\in J_h} \in \mathcal{S}_h\}$ is proposed in [4], but it requires to solve generally more than one LP with feasible set $\mathcal{S}_h$. The problem has been also examined from a geometrical point of view in [12]. We will see that it is possible to find a solution that minimises the number of elements of I_{h+1} by solving only one LP.

Let n_h be the number of elements of I_h and m_h the number of elements of J_h. Then system (S_h) has the form $D^h x = d^h, x \geq 0$, where D^h is a matrix of $n_h + 1$ rows and m_h columns and d^h a vector of $\Re^{(n_h+1)}$ properly defined. We will indicate with (MNS_h) the linear programming problem of minimisation of the null function over $\mathcal{S}_h$.

Let (S_h) be consistent and suppose we discover which indexes correspond to implied equalities among the inequalities $x_j \geq 0$ $(j \in J_h)$ of (S_h). We know from Prop. 4.2 that there exists a solution $x^*(h)$ of (S_h) such that $\{1, \ldots, m_h\} - \sigma(x^*(h))$ is the set of such indexes and we can resort to Prop. 4.3 or Prop. 4.4 to determine it (by setting $A' = D^h$ and $b' = d^h$). Indeed, observe that both propositions let us to determine $x^*(h)$ directly, but what really matters is $\sigma(x^*(h))$.

It is $\sigma(x(h)) \subseteq \sigma(x^*(h))$ for every solution $x(h)$ of (S_h). Thus, if we choose the solution $x^*(h)$, we reduce the number of indexes contained in I_{h+1} and the number of atoms of J_{h+1} as much as possible. Indeed, with this choice

$$i \in I_{h+1} \text{ iff } \sum_{j \in J_h} x_j(h)\delta(e_j \Rightarrow H_i) = 0 \ \forall (x_j(h))_{j \in J_h} \in S_h$$

and therefore it is

$$I_{h+1} = \{i \in I_h : \max \sum_{j \in J_h} x_j(h)\delta(e_j \Rightarrow H_i) = 0, (x_j(h))_{j \in J_h} \in S_h\},$$

so that only the systems that are strictly necessary to the algorithm have to be examined.

Observe that, by applying Prop. 4.3, it is necessary to determine a strictly complementary solution of (MNS_h). For this purpose, as illustrated in Sect. 3, we can use an interior point method or also, if we prefer, resort to Prop. 3.6 and use the simplex algorithm to solve the corresponding problem $(LPSC)$. In this case, if we detect the infeasibility of $(LPSC)$, we can deduce the inconsistency of (S_h) anyhow, because the primal in Prop. 4.3 is always feasible. On the other hand, Prop. 4.4 relies on a problem directly solvable by means of the simplex algorithm, even if no one prevents us from solving it by resorting to an interior point method.

Notice that, by applying Prop. 3.6 or Prop. 4.4, the number of rows and of columns of the systems considered increases appreciably. This observation could suggest us to apply Prop. 4.3 and an interior point method, when the size of the system (S_h) is large, because of the very good performance generally attained by these methods when applied to large scale linear programming problems.

Remarks Let P be coherent and denote with $\mathcal{Q}$ the set of the coherent probabilities extending P to $\mathcal{A}_L(\mathcal{P}|K_1)$. In this hypothesis, every probability P' of $\mathcal{Q}$ is determined by setting, for every solution $(x_j(1))_{j \in J_1}$ of (S_1), $P'(e_j|K_1) = x_j(1)$ and then $P'(E|K_1) = \sum_{j \in J_1} x_j(1)\delta(e_j|K_1 \Rightarrow E|K_1) = \sum_{j \in J_1} x_j(1)\delta(e_j \Rightarrow E)$ for every $E|K_1 \in \mathcal{A}_L(\mathcal{P}|K_1)$ ([25]). Let $x^*(1)$ be a solution of (S_1) (determined by means of Prop. 4.3 or Prop. 4.4) such that $x_j \geq 0$ is an implied equality iff $j \notin \sigma(x^*(1))$. Then, it is $P'(e_j|K_1) = 0 \ \forall P' \in \mathcal{Q}$ iff $j \notin \sigma(x^*(1))$. Thus, $\{1, ..., m_h\} - \sigma(x^*(1))$ is the set of indexes of the atoms of the partition $\mathcal{P}|K_1$ that have null probability for each coherent extension of P to $\mathcal{A}_L(\mathcal{P}|K_1)$. Besides, by choosing $x^*(1)$ as solution of (S_1), it is $I_2 = \{i : P'(H_i|K_1) = 0 \ \forall P' \in \mathcal{Q}\}$.

Therefore, we can identify the events H_i such that $P'(H_i|K_1) = 0$ for every P' extending P to $\mathcal{A}_L(\mathcal{P}|K_1)$ by solving one LP. Other procedures that require to solve generally more linear programming problems for identifying these events are described in [11, 25].

Consider an event $E|H \notin \mathcal{E}$ and suppose P coherent. It is well known that P can be extended to a coherent probability P^* on $\mathcal{E} \cup \{E|H\}$ ([16]). Besides, the set of admissible values of $P^*(E|H)$ is a closed (non empty) interval $[P_L(E|H), P_U(E|H)]$. Strict complementarity can be applied also in implementing the algorithm that determines the endpoints of this interval proposed in [26].

Suppose $E|H \in \mathcal{A}_L(\mathcal{P})|\mathcal{A}_L^{\phi}(\mathcal{P})$ (it can be shown that this assumption is not restrictive). We report the algorithm that calculates $P_L(E|H)$. The same algorithm can be used to calculate $P_U(E|H)$, because it is $P_U(E|H) = 1 - P_L(\overline{E}|H)$ (see [26]).

1) Set h:=1.
2) If $\overline{E} \wedge H \wedge \overline{K}_h \neq \phi$ then $P_L(E|H) = 0$; stop
3) Consider the following linear programming problem (LPS_h)

$$(S_h) \qquad \begin{aligned} & m_h = \min \sum_{j \in J_h} x_j \delta(e_j \Rightarrow H \wedge K_h) \\ & \sum_{j \in J_h} x_j \delta(e_j \Rightarrow E_i \wedge H_i) - p_i \sum_{j \in J_h} x_j \delta(e_j \Rightarrow H_i) = 0 \quad (i \in I_h) \\ & \sum_{j \in J_h} x_j = 1, \ x_j \geq 0 \ (j \in J_h) \end{aligned}$$

If $m_h > 0$ then it is

$$\begin{aligned} & P_L(E|H) = \min \sum_{j \in J_h} x_j \delta(e_j \Rightarrow E \wedge H \wedge K_h) \\ & \sum_{j \in J_h} x_j \delta(e_j \Rightarrow E_i \wedge H_i) - p_i \sum_{j \in J_h} x_j \delta(e_j \Rightarrow H_i) = 0 \quad (i \in I_h) \\ & \sum_{j \in J_h} x_j \delta(e_j \Rightarrow H \wedge K_h) = 1, x_j \geq 0 \ (j \in J_h) \end{aligned}$$

stop.
4) Let $(x(h))_{j \in J_h}$ be a solution of the problem (LPS_h).
5) Define
$I_{h+1} = \{i \in I_h : \sum_{j \in J_h} x_j(h) \delta(e_j \Rightarrow H_i) = 0\}$,
$K_{h+1} = \vee_{i \in I_{h+1}} H_i$,

$J_{h+1} = \{j \in J_h : e_j \Rightarrow K_{h+1}\}$.

6) Set h:=h+1.
7) Goto 2).

In this case I_{h+1} and J_{h+1} depend on the solution found for the linear programming problem (LPS_h). Observe that S_h is compact and it is non-empty, owing to the coherence of P, so that in any case (LPS_h) has solution. Let $x^*(h)$ be a strictly complementary solution of (LPS_h). Then, by applying Cor. 3.5 with $A^T = D^h$, $c = d^h$ and $b = (\delta(e_j \Rightarrow H \wedge K_h))_{j \in J_h}$, it is $\sigma(x(h)) \subseteq \sigma(x^*(h))$ for every solution $x(h)$ of (LPS_h). Thus, the choice of $x^*(h)$ among all the solutions of (LPS_h) determines the smallest I_{h+1} and J_{h+1} possible. As above, we can find such a solution by means of an interior point method or, if we prefer, by resorting to Prop. 3.6. We just observe that an analogue technique can be applied also to the problem of determining the least-committal correction of an assigned avoiding sure loss lower probability on a finite set of conditional events and refer to [22] for a description of the corresponding algorithm.

6. An example.

In this section we exemplify the use of the strict complementarity in checking coherence. In order to perform the calculation, we have realised a program. Given a finite set of events and a probabilistic assignment on it, this program firstly determines the atoms of the corresponding partition, by using the algorithm presented in [3] with some slight modifications, and then checks the coherence of this assignment by means of the algorithm previously recalled. The program language used is Maple V, owing of its capacity of performing symbolic computation. In the numerical results that follow, only the first six decimal significant digits are reported.

Let the set of events be

$$\mathcal{E} = \{E_1 \vee E_2|H_1, E_1|H_1, E_2|H_1, E_1 \wedge E_2|H_2, E_1 \vee E_2|H_3, E_2 \vee E_3|H_2, E_3|H_1\}$$

and suppose the following logical relations among the events are known

$$\begin{array}{lll} E_1 \wedge E_2 \wedge \overline{H}_1 = \phi & H_1 \wedge \overline{H}_2 = \phi & E_3 \wedge \overline{E}_1 = \phi \\ E_3 \wedge \overline{H}_2 = \phi & E_3 \wedge \overline{H}_1 = \phi & E_3 \wedge \overline{E}_2 = \phi \\ E_3 \wedge \overline{H}_3 = \phi & E_2 \wedge \overline{H}_3 = \phi & \end{array}$$

Let the following assessment on $\mathcal{E}$ be assigned:

$$
\begin{array}{lll}
P(E_1 \vee E_2|H_1) = 0.7 & P(E_1|H_1) = 0.6 & P(E_2|H_1) = 0.3 \\
P(E_1 \wedge E_2|H_2) = 0.1 & P(E_1 \vee E_2|H_3) = 0.7 & P(E_2 \vee E_3|H_2) = 0.3 \\
P(E_3|H_1) = 0.2 & &
\end{array}
$$

The atoms of partition $\mathcal{P}$ are the following

$$
\begin{array}{ll}
e_1 = (E_1 \vee E_2) \wedge \overline{H}_1 \wedge \overline{H}_2 \wedge H_3 & e_2 = \overline{E}_1 \wedge \overline{E}_2 \wedge \overline{H}_1 \wedge \overline{H}_2 \wedge H_3 \\
e_3 = \overline{E}_1 \wedge E_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge H_3 & e_4 = \overline{E}_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge \overline{H}_1 \wedge H_2 \wedge H_3 \\
e_5 = E_1 \wedge E_2 \wedge E_3 \wedge H_1 \wedge H_2 \wedge H_3 & e_6 = E_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge \overline{H}_1 \wedge H_2 \wedge H_3 \\
e_7 = \overline{E}_2 \wedge \overline{E}_3 \wedge \overline{H}_1 \wedge H_2 \wedge \overline{H}_3 & e_8 = ((E_1 \wedge E_2) \vee (E_1 \wedge \overline{E}_2)) \wedge \overline{H}_1 \wedge H_2 \wedge H_3 \\
e_9 = \overline{E}_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge H_3 & e_{10} = E_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge H_3 \\
e_{11} = \overline{E}_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge \overline{H}_3 & e_{12} = E_1 \wedge \overline{E}_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge \overline{H}_3 \\
e_{13} = E_1 \wedge E_2 \wedge \overline{E}_3 \wedge H_1 \wedge H_2 \wedge H_3 &
\end{array}
$$

System (S_1) is then

$$
\begin{cases}
0.3x_3 + 0.3x_5 - 0.7x_9 + 0.3x_{10} - 0.7x_{11} + 0.3x_{12} + 0.3x_{13} = 0 \\
-0.6x_3 + 0.4x_5 - 0.6x_9 + 0.4x_{10} - 0.6x_{11} + 0.4x_{12} + 0.4x_{13} = 0 \\
0.7x_3 + 0.7x_5 - 0.3x_9 - 0.3x_{10} - 0.3x_{11} - 0.3x_{12} + 0.7x_{13} = 0 \\
-0.1x_3 - 0.1x_4 + 0.9x_5 - 0.1x_6 - 0.1x_7 - 0.1x_8 - 0.1x_9 - 0.1x_{10} - 0.1x_{11} + \\
-0.1x_{12} + 0.9x_{13} = 0 \\
0.3x_1 - 0.7x_2 + 0.3x_3 - 0.7x_4 + 0.3x_5 + 0.3x_6 + 0.3x_8 - 0.7x_9 - 0.3x_{10} + 0.3x_{13} = 0 \\
0.7x_3 - 0.3x_4 + 0.7x_5 - 0.3x_6 - 0.3x_7 + 0.7x_8 - 0.3x_9 - 0.3x_{10} - 0.3x_{11} + \\
-0.3x_{12} + 0.7x_{13} = 0 \\
-0.2x_3 + 0.8x_5 - 0.2x_9 - 0.2x_{10} - 0.2x_{11} - 0.2x_{12} - 0.2x_{13} = 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} = 1 \\
x_j \geq 0 \ (j = 1, \ldots, 13)
\end{cases}
$$

By applying the simplex algorithm to MNS_1, we have found the solution

$$x_1 = 0.7, x_2 = 0.3, x_j = 0 \ (j = 3, \ldots, 13).$$

So it is $I_2 = \{1, 2, 3, 4, 6, 7\}$ and $J_2 = \{3, \ldots, 13\}$. Then, the system (S_2) is determined as follows

$$\begin{cases} 0.3x_3 + 0.3x_5 - 0.7x_9 + 0.3x_{10} - 0.7x_{11} + 0.3x_{12} + 0.3x_{13} = 0 \\ -0.6x_3 + 0.4x_5 - 0.6x_9 + 0.4x_{10} - 0.6x_{11} + 0.4x_{12} + 0.4x_{13} = 0 \\ 0.7x_3 + 0.7x_5 - 0.3x_9 - 0.3x_{10} - 0.3x_{11} - 0.3x_{12} + 0.7x_{13} = 0 \\ -0.1x_3 - 0.1x_4 + 0.9x_5 - 0.1x_6 - 0.1x_7 - 0.1x_8 - 0.1x_9 - 0.1x_{10} - 0.1x_{11} + \\ -0.1x_{12} + 0.9x_{13} = 0 \\ 0.7x_3 - 0.3x_4 + 0.7x_5 - 0.3x_6 - 0.3x_7 + 0.7x_8 - 0.3x_9 - 0.3x_{10} - 0.3x_{11} + \\ -0.3x_{12} + 0.7x_{13} = 0 \\ -0.2x_3 + 0.8x_5 - 0.2x_9 - 0.2x_{10} - 0.2x_{11} - 0.2x_{12} - 0.2x_{13} = 0 \\ x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} = 1 \\ x_j \geq 0 \ (j = 3, \ldots, 13) \end{cases}$$

With the simplex algorithm again, we have found the solution

$$x_3 = 0.05, x_4 = 0.35, x_5 = 0.1, x_8 = x_{11} = 0.15, x_{12} = 0.2, x_j = 0 \ (j = 6, 7, 9, 10, 13).$$

It is $I_3 = \emptyset$ and so P is coherent.

Therefore, in order to verify the coherence of P, we have examined two systems. Then, as suggested by Prop. 4.3, we have determined a strictly complementary optimal solution of (MNS_1). By resorting to Prop. 3.6 and to the simplex algorithm, the solution found is

$$x_4 = 0.123377, x_7 = 0.149351, x_8 = x_{10} = 0.136364, x_5 = x_{11} = 0.090909, x_{13} = 0,$$

$$x_j = 0.045455 \ (j = 1, 2, 3, 6, 9, 12).$$

This solution tells us $x_{13} \geq 0$ is an implied equality, that is $x_{13} = 0 \ \forall x$ solution of (S_1). Thus, we obtain $I_2 = \emptyset$ and we can affirm directly that P is coherent. Besides, we know that $P^*(e_{13}|K_1) = 0$ for each coherent probability P^* extending P to $\mathcal{A}_L(\mathcal{P}|K_1)$, where $K_1 = H_1 \vee H_2 \vee H_3$.

We have also solved (MNS_1) by means of an interior point method. Among the various existing interior point algorithms, we have chosen the algorithm proposed in [28]. It is based on the construction of a homogeneous and self-dual artificial LP embedding the problem to be examined and its corresponding dual. This algorithm is exempt from some drawbacks present in other interior point algorithms proposed in literature (see [1] for a comment on its properties), but presents a deficiency with respect to the simplex: it correctly detects the infeasibility for at least one between the primal and the dual, but not necessarily for both. This shortcoming does not affect our application, because in Prop. 4.3 the primal is always feasible. We have used the Matlab code freely provided by Y.Ye at http://dollar.biz.uiowa.edu/col/ye/matlab.html. By applying this algorithm, we have obtained the following solution of (S_1)

$$x_1 = 0.105226, x_2 = 0.057058, x_3 = 0.041886, x_4 = 0.066557, x_5 = 0.083772$$

$$x_6 = 0.124274, x_7 = 0.102369, x_8 = 0.125657, x_9 = 0.049821, x_{10} = 0.076131$$

$$x_{11} = 0.075836, x_{12} = 0.091412, x_{13} = 0$$

which confirms that $I_2 = \emptyset$ and P is coherent.

We have seen that, by applying the strict complementarity, only one system has been examined to verify the coherence of P. For the sake of completeness, we observe that, in this example, by applying the simplex to (MNS_1), it could happen to find a basic solution such that $I_2 = \emptyset$. For instance,

$$x_1 = 0.464286, x_3 = 0.026786, x_4 = 0.1875, x_5 = 0.053571, x_8 = x_9 = 0.080357$$

$$x_{12} = 0.107143, x_j = 0 \ (j = 2, 6, 7, 10, 11, 13)$$

is also a basic solution of (MNS_1). It does not determine the implied equalities of (S_1), but it enable us to conclude directly that $I_2 = \emptyset$ and that P is coherent. However, in general, as we have previously seen, a solution of (S_h) found directly by means of the simplex does not guarantee us to minimise the number of the elements of I_{h+1} and J_{h+1}, whereas a strictly complementary solution of (MNS_h) does.

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