Least Integral-Squared Error Design of FIR Filters

- Let $H_d(e^{j\omega})$ denote the desired frequency response
- Since $H_d(e^{j\omega})$ is a periodic function of ω with a period 2π, it can be expressed as a Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n] e^{-j\omega n}$$

where

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} \, d\omega, \quad -\infty < n < \infty$$

Least Integral-Squared Error Design of FIR Filters

- In general, $H_d(e^{j\omega})$ is piecewise constant with sharp transitions between bands
- In which case, $\{h_d[n]\}$ is of infinite length and noncausal
- Objective - Find a finite-duration $\{h_t[n]\}$ of length $2M+1$ whose DTFT $H_t(e^{j\omega})$ approximates the desired DTFT $H_d(e^{j\omega})$ in some sense

Least Integral-Squared Error Design of FIR Filters

- Commonly used approximation criterion - Minimize the integral-squared error

$$\Phi = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| H_t(e^{j\omega}) - H_d(e^{j\omega}) \right|^2 d\omega$$

where

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n] e^{-j\omega n}$$

Least Integral-Squared Error Design of FIR Filters

- Using Parseval's relation we can write

$$\Phi = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

$$= \sum_{n=-M}^{M} \left| h_t[n] - h_d[n] \right|^2 + \sum_{n=-\infty}^{-M-1} h_d^2[n] + \sum_{n=M+1}^{\infty} h_d^2[n]$$

- It follows from the above that Φ is minimum when $\boxed{h_t[n] = h_d[n] \text{ for } -M \leq n \leq M}$
- $\Rightarrow$ Best finite-length approximation to ideal infinite-length impulse response in the mean-square sense is obtained by truncation

Least Integral-Squared Error Design of FIR Filters

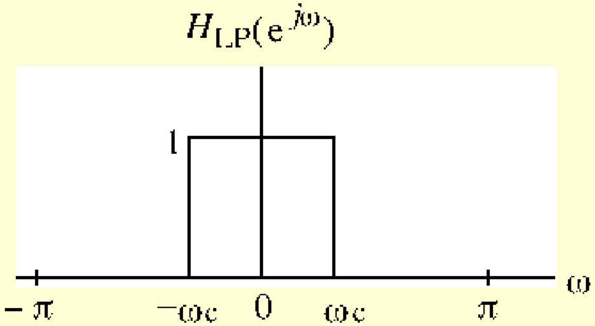
- A causal FIR filter with an impulse response $h[n]$ can be derived from $h_t[n]$ by delaying:

$$h[n] = h_t[n-M]$$

- The causal FIR filter $h[n]$ has the same magnitude response as $h_t[n]$ and its phase response has a linear phase shift of ωM radians with respect to that of $h_t[n]$

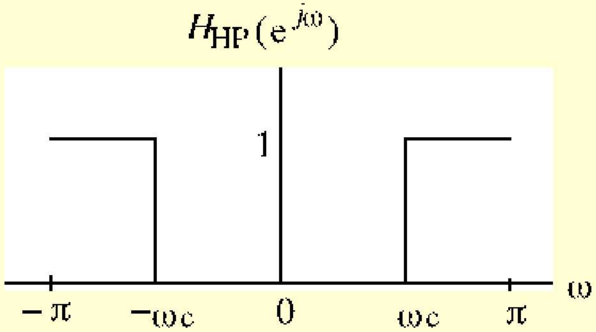
Impulse Responses of Ideal Filters

- Ideal lowpass filter -



$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

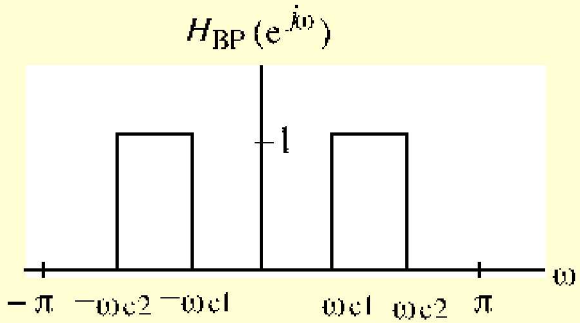
- Ideal highpass filter -



$$h_{HP}[n] = \begin{cases} 1 - \frac{\omega_c}{\pi}, & n = 0 \\ -\frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

Impulse Responses of Ideal Filters

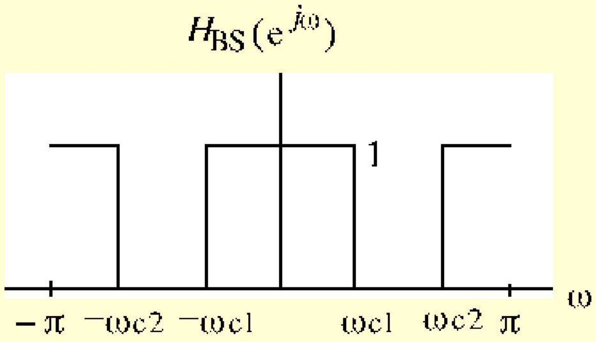
- Ideal bandpass filter -



$$h_{BP}[n] = \begin{cases} \dfrac{\sin(\omega_{c2} n)}{\pi n} - \dfrac{\sin(\omega_{c1} n)}{\pi n}, & n \neq 0 \\ \dfrac{\omega_{c2}}{\pi} - \dfrac{\omega_{c1}}{\pi}, & n = 0 \end{cases}$$

Impulse Responses of Ideal Filters

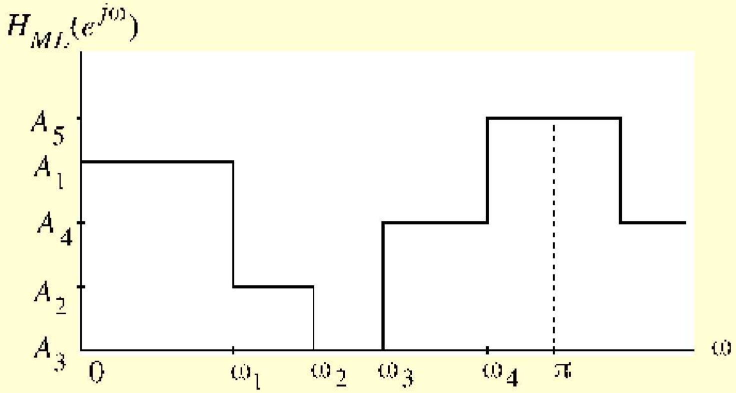
- Ideal bandstop filter -



$$h_{BS}[n] = \begin{cases} 1 - \dfrac{(\omega_{c2} - \omega_{c1})}{\pi}, & n = 0 \\ \dfrac{\sin(\omega_{c1} n)}{\pi n} - \dfrac{\sin(\omega_{c2} n)}{\pi n}, & n \neq 0 \end{cases}$$

Impulse Responses of Ideal Filters

- Ideal multiband filter -



$$H_{ML}(e^{j\omega}) = A_k,$$

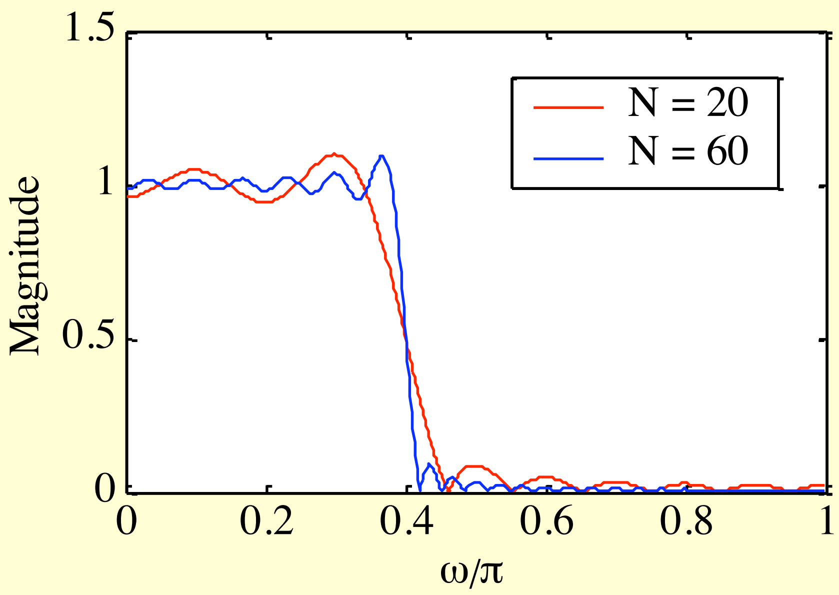
$$\omega_{k-1} \le \omega \le \omega_k,$$

$$k = 1, 2, \ldots, L$$

$$h_{ML}[n] = \sum_{\ell=1}^{L} (A_\ell - A_{\ell+1}) \cdot \frac{\sin(\omega_L n)}{\pi n}$$

Gibbs Phenomenon

- Gibbs phenomenon - Oscillatory behavior in the magnitude responses of causal FIR filters obtained by truncating the impulse response coefficients of ideal filters



see slides 3.1.x about mean-square convergence of the DTFT

Gibbs Phenomenon

- Gibbs phenomenon can be explained by treating the truncation operation as an windowing operation:

$$h_t[n] = h_d[n] \cdot w[n]$$

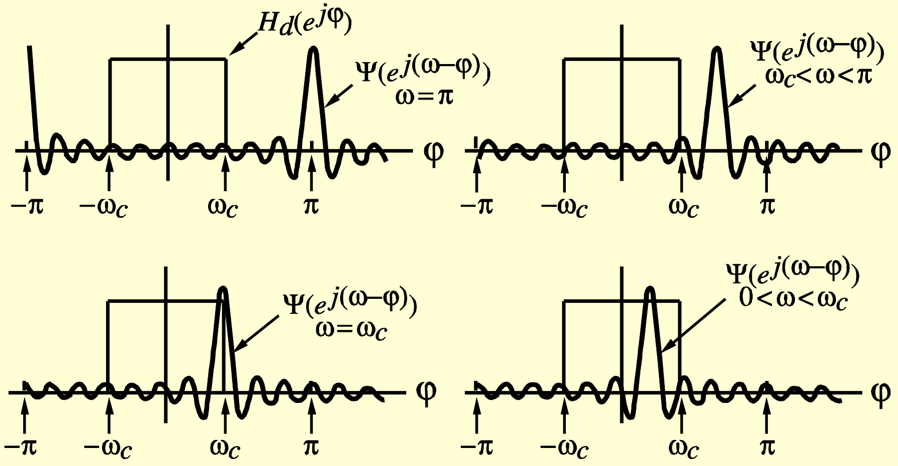
- In the frequency domain

$$H_t(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\varphi}) \Psi(e^{j(\omega-\varphi)})\, d\varphi$$

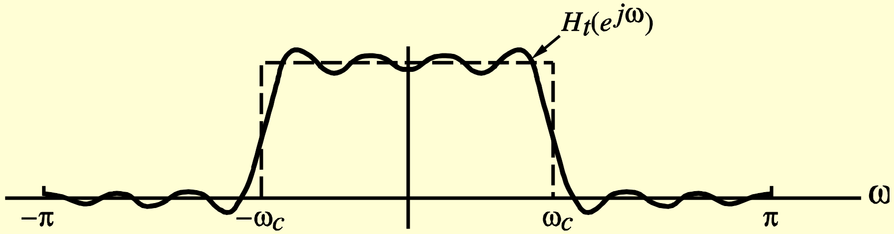
- where $H_t(e^{j\omega})$ and $\Psi(e^{j\omega})$ are the DTFTs of $h_t[n]$ and $w[n]$, respectively

Gibbs Phenomenon

- Thus $H_t(e^{j\omega})$ is obtained by a periodic continuous convolution of $H_d(e^{j\omega})$ with $\Psi(e^{j\omega})$:



(a)



(b)

Gibbs Phenomenon

- If $\Psi(e^{j\omega})$ is a very narrow pulse centered at $\omega = 0$ (ideally a delta function) compared to variations in $H_d(e^{j\omega})$, then $H_t(e^{j\omega})$ will approximate $H_d(e^{j\omega})$ very closely
- Length $2M+1$ of $w[n]$ should be very large
- On the other hand, length $2M+1$ of $h_t[n]$ should be as small as possible to reduce computational complexity

Gibbs Phenomenon

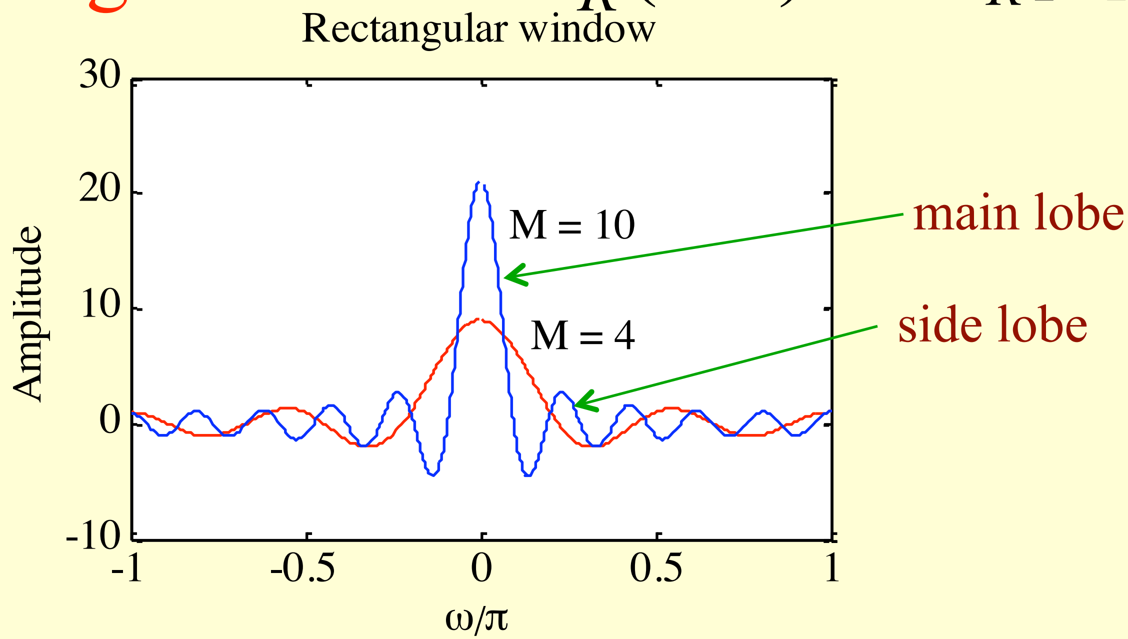
- A **rectangular window** is used to achieve simple truncation:

$$w_R[n] = \begin{cases} 1, & 0 \le |n| \le M \\ 0, & \text{otherwise} \end{cases}$$

- Presence of oscillatory behavior in $H_t(e^{j\omega})$ is basically due to:
  - 1) $h_d[n]$ is infinitely long and not absolutely summable, and hence filter is unstable
  - 2) Rectangular window has an abrupt transition to zero

Gibbs Phenomenon

- Oscillatory behavior can be explained by examining the DTFT $\Psi_R(e^{j\omega})$ of $w_R[n]$:



- $\Psi_R(e^{j\omega})$ has a **main lobe** centered at $\omega = 0$
- Other ripples are called **sidelobes**

Gibbs Phenomenon

- Rectangular window has an abrupt transition to zero outside the range $-M \leq n \leq M$, which results in Gibbs phenomenon in $H_t(e^{j\omega})$
- Gibbs phenomenon can be reduced either:

(1) Using a window that tapers smoothly to zero at each end, or

(2) Providing a smooth transition from passband to stopband in the magnitude specifications

Fixed Window Functions

- Using a tapered window causes the height of the sidelobes to diminish, with a corresponding increase in the main lobe width resulting in a wider transition at the discontinuity

"raised cosine" windows

- Hann:

$$w[n] = 0.5 + 0.5\cos(\frac{\pi n}{M}), \qquad -M \le n \le M$$

- Hamming:

$$w[n] = 0.54 + 0.46\cos(\frac{\pi n}{M}), \quad -M \le n \le M$$

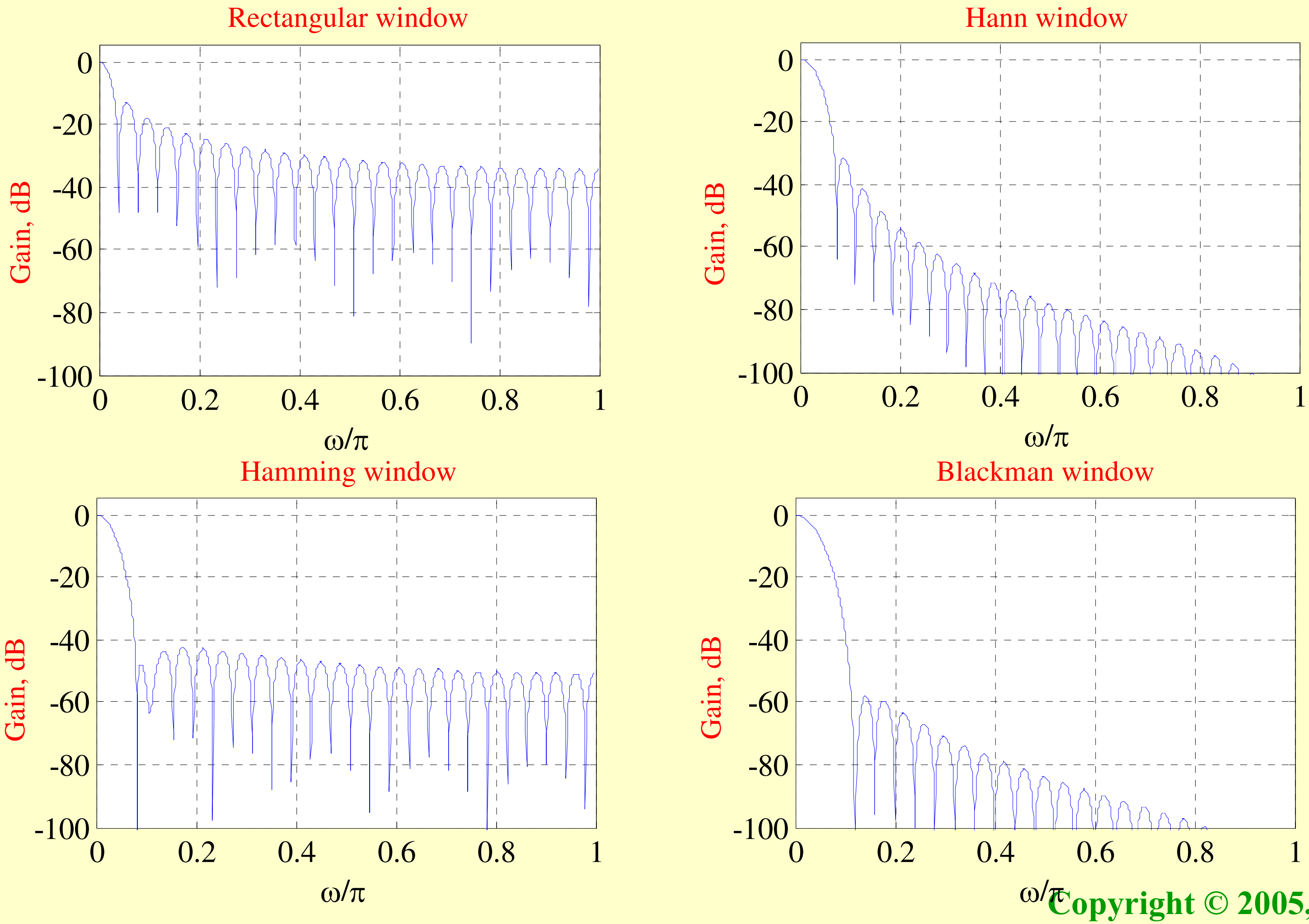
- Blackman:

$$w[n] = 0.42 + 0.5\cos(\frac{\pi n}{M}) + 0.08\cos(\frac{2\pi n}{M})$$

$$-M \le n \le M$$

Fixed Window Functions

- Plots of magnitudes of the DTFTs of these windows for $M = 25$ are shown below:



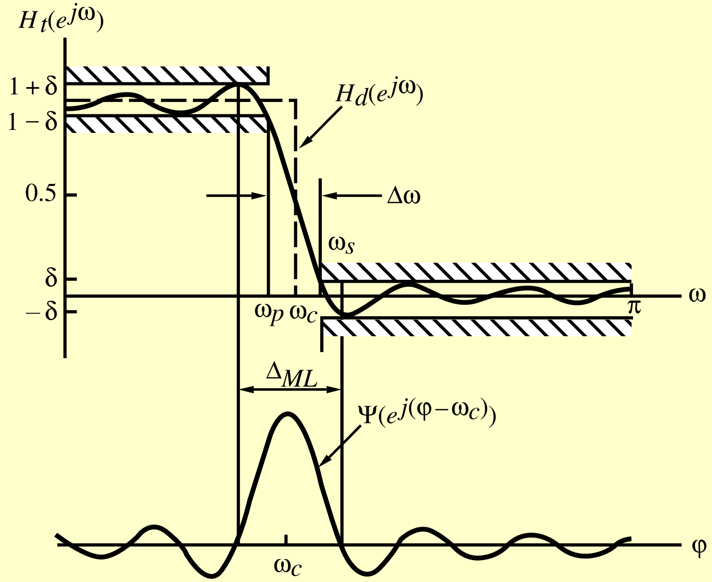
Fixed Window Functions

- Magnitude spectrum of each window characterized by a main lobe centered at $\omega = 0$ followed by a series of sidelobes with decreasing amplitudes
- Parameters predicting the performance of a window in filter design are:
- **Main lobe width**
- **Relative sidelobe level**

Fixed Window Functions

- $\boxed{\textbf{Main lobe width } \Delta_{ML}}$ - given by the distance between zero crossings on both sides of main lobe
- $\boxed{\textbf{Relative sidelobe level } A_{s\ell}}$ - given by the difference in dB between amplitudes of largest sidelobe and main lobe

Fixed Window Functions



- Observe $H_t(e^{j(\omega_c+\Delta\omega)}) + H_t(e^{j(\omega_c-\Delta\omega)}) \cong 1$
- Thus, $H_t(e^{j\omega_c}) \cong 0.5$
- Passband and stopband ripples are the same

Fixed Window Functions

- Distance between the locations of the maximum passband deviation and minimum stopband value $\cong \Delta_{ML}$

- Width of transition band

$$\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$$

Fixed Window Functions

- To ensure a fast transition from passband to stopband, window should have a very small main lobe width
- To reduce the passband and stopband ripple δ, the area under the sidelobes should be very small
- Unfortunately, these two requirements are contradictory

Fixed Window Functions

- In the case of rectangular, Hann, Hamming, and Blackman windows, the value of ripple does not depend on filter length or cutoff frequency ω_c, and is essentially constant
- In addition,

$$\Delta\omega \approx \frac{c}{M}$$

where c is a constant for most practical purposes

Fixed Window Functions

- **Rectangular window** - $\Delta_{ML} = 4\pi/(2M+1)$

  $A_{s\ell} = 13.3$ dB, $\alpha_s = 20.9$ dB, $\Delta\omega = 0.92\pi/M$

- **Hann window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 31.5$ dB, $\alpha_s = 43.9$ dB, $\Delta\omega = 3.11\pi/M$

- **Hamming window** - $\Delta_{ML} = 8\pi/(2M+1)$

  $A_{s\ell} = 42.7$ dB, $\alpha_s = 54.5$ dB, $\Delta\omega = 3.32\pi/M$

- **Blackman window** - $\Delta_{ML} = 12\pi/(2M+1)$

  $A_{s\ell} = 58.1$ dB, $\alpha_s = 75.3$ dB, $\Delta\omega = 5.56\pi/M$

Fixed Window Functions

- Filter Design Steps -

(1) Set

$$\omega_c = (\omega_p + \omega_s)/2$$

(2) Choose window based on specified α_s

(3) Estimate M using

$$\Delta\omega \approx \frac{c}{M}$$

FIR Filter Design Example

- Lowpass filter of length 51 and $\omega_c = \pi/2$