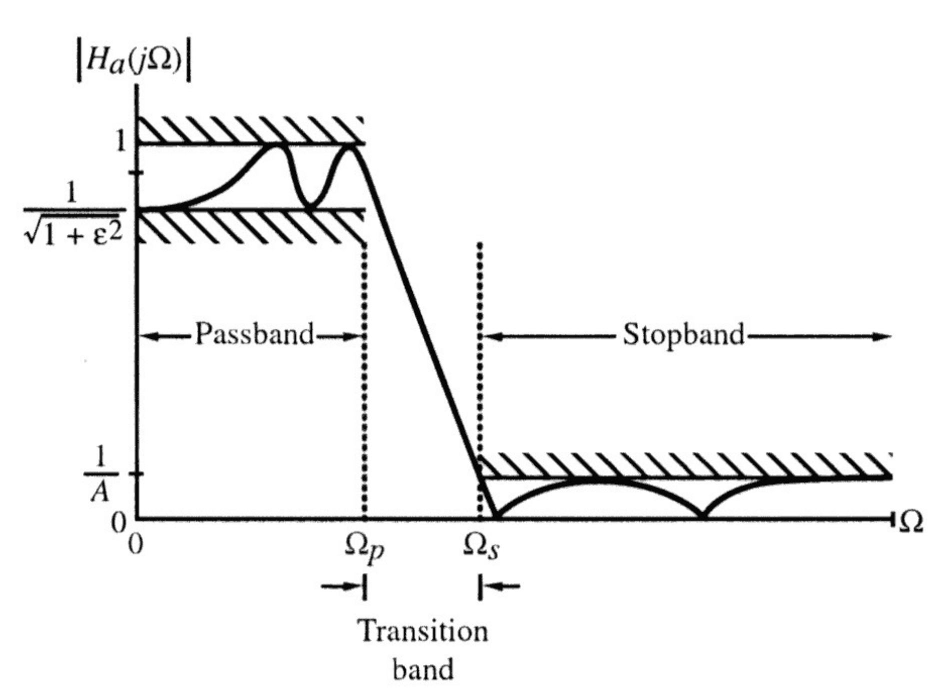
Analog Lowpass Filter Specifications

- Magnitude specifications may alternately be given in a normalized form as indicated below



Analog Lowpass Filter Specifications

- Here, the maximum value of the magnitude in the passband assumed to be unity
- $1/\sqrt{1+\varepsilon^2}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband
- $\frac{1}{A}$ - Maximum stopband magnitude

Analog Lowpass Filter Design

- Two additional parameters are defined -

(1) **Transition ratio** $k = \dfrac{\Omega_p}{\Omega_s}$

For a lowpass filter $k < 1$

(2) **Discrimination parameter** $k_1 = \dfrac{\varepsilon}{\sqrt{A^2 - 1}}$

Usually $k_1 << 1$

Butterworth Approximation

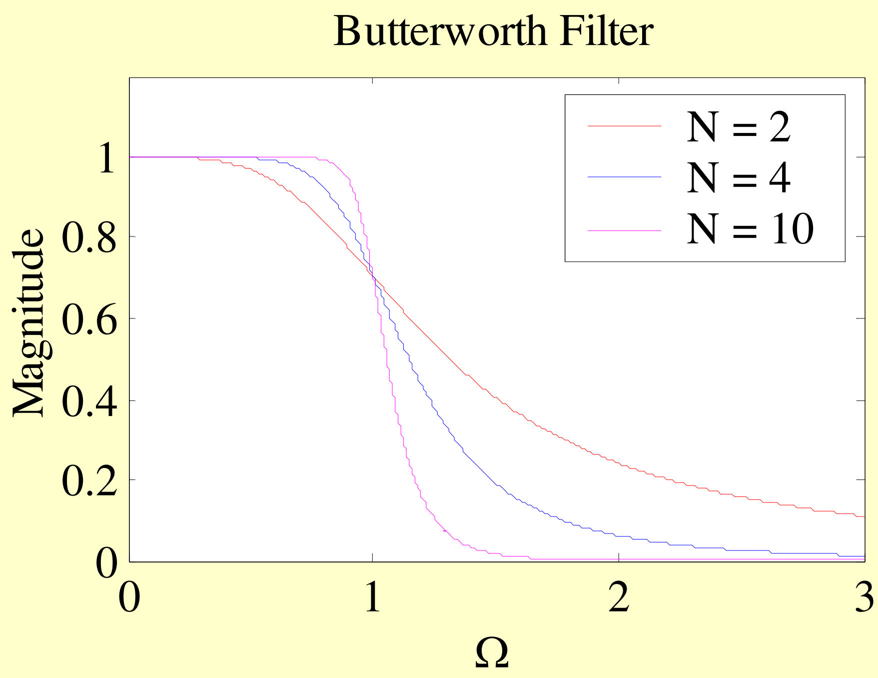
- The magnitude-square response of an N-th order analog lowpass **Butterworth filter** is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

- First $2N-1$ derivatives of $|H_a(j\Omega)|^2$ at $\Omega = 0$ are equal to zero
- The Butterworth lowpass filter thus is said to have a **maximally-flat magnitude** at $\Omega = 0$

Butterworth Approximation

- Typical magnitude responses with $\Omega_c = 1$



Butterworth Approximation

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N
- These are determined from the specified bandedges Ω_p and Ω_s, and minimum passband magnitude $1/\sqrt{1+\varepsilon^2}$, and maximum stopband ripple $1/A$

Butterworth Approximation

- Ω_c and N are thus determined from

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{1}{2}\cdot\frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)} = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)}$$

Butterworth Approximation

- Since order N must be an integer, value obtained is rounded up to the next highest integer
- This value of N is used next to determine Ω_c by satisfying either the stopband edge or the passband edge specification exactly
- If the stopband edge specification is satisfied, then the passband edge specification is exceeded providing a safety margin

Butterworth Approximation

- Transfer function of an analog Butterworth lowpass filter is given by

$$H_a(s) = \frac{C}{D_N(s)} = \frac{\Omega_c^N}{s^N + \sum_{\ell=0}^{N-1} d_\ell s^\ell} = \frac{\Omega_c^N}{\prod_{\ell=1}^{N}(s - p_\ell)}$$

where

$$p_\ell = \Omega_c e^{j[\pi(N+2\ell-1)/2N]}, \ 1 \leq \ell \leq N$$

- Denominator $D_N(s)$ is known as the **Butterworth polynomial** of order N

Butterworth Approximation

- Example - Determine the lowest order of a Butterworth lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz
- Now

$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) = -1$$

which yields $\varepsilon^2 = 0.25895$

and

$$10\log_{10}\left(\frac{1}{A^2}\right) = -40$$

which yields $A^2 = 10{,}000$

14

Butterworth Approximation

- Therefore $\dfrac{1}{k_1} = \dfrac{\sqrt{A^2 - 1}}{\varepsilon} = 196.51334$

  and $\dfrac{1}{k} = \dfrac{\Omega_s}{\Omega_p} = 5$

- Hence

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.2811$$

- We choose $N = 4$

Chebyshev Approximation

- The magnitude-square response of an *N*-th order analog lowpass **Type 1 Chebyshev filter** is given by

$$|H_a(s)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

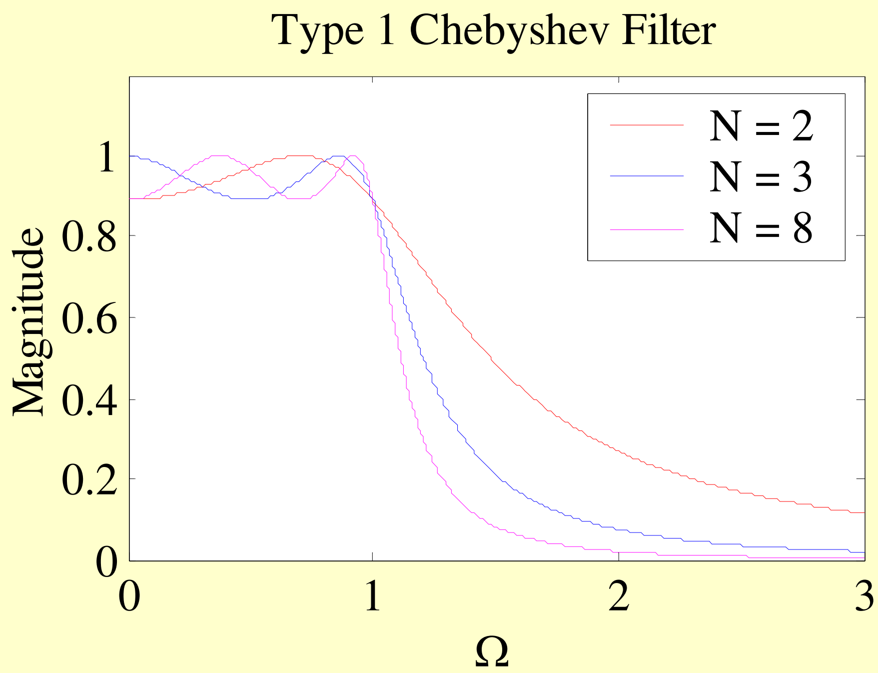
where $T_N(\Omega)$ is the **Chebyshev polynomial** of order *N*:

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases}$$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$$

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 1 Chebyshev filter are shown below



Chebyshev Approximation

- If at $\Omega = \Omega_s$ the magnitude is equal to $1/A$, then

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Order N is chosen as the nearest integer greater than or equal to the above value

Chebyshev Approximation

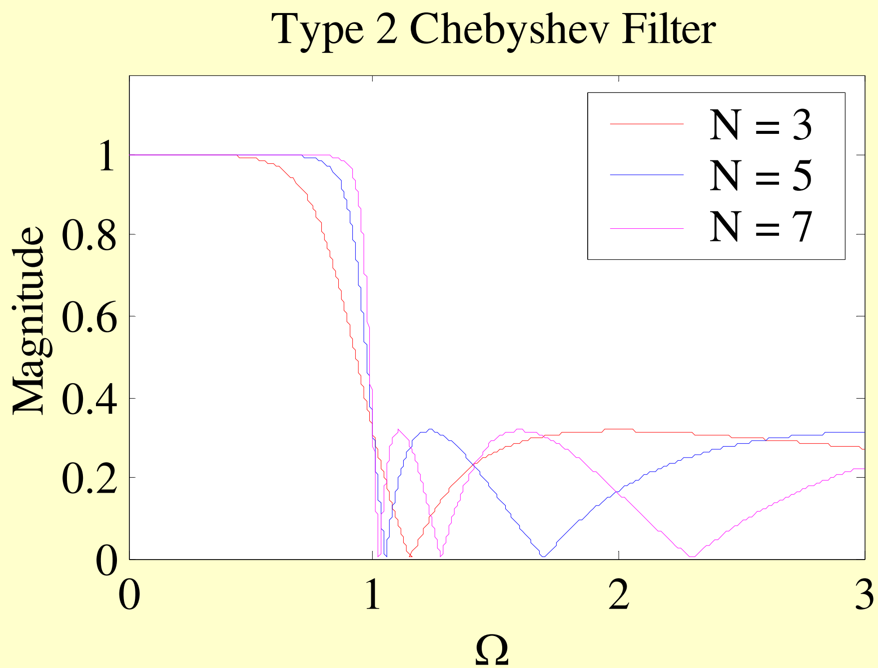
- The magnitude-square response of an N-th order analog lowpass **Type 2 Chebyshev** (also called **inverse Chebyshev**) **filter** is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}\right]^2}$$

where $T_N(\Omega)$ is the **Chebyshev polynomial** of order N

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 2 Chebyshev filter are shown below



Chebyshev Approximation

- The order N of the Type 2 Chebyshev filter is determined from given ε, Ω_s, and A using

$$N = \frac{\cosh^{-1}(\sqrt{A^2 - 1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Example - Determine the lowest order of a Chebyshev lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz -

$$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.6059$$

21

Elliptic Approximation

- The square-magnitude response of an elliptic lowpass filter is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

where $R_N(\Omega)$ is a rational function of order N satisfying $R_N(1/\Omega) = 1/R_N(\Omega)$, with the roots of its numerator lying in the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$

Elliptic Approximation

- For given Ω_p, Ω_s, ε, and A, the filter order can be estimated using

$$N \cong \frac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)}$$

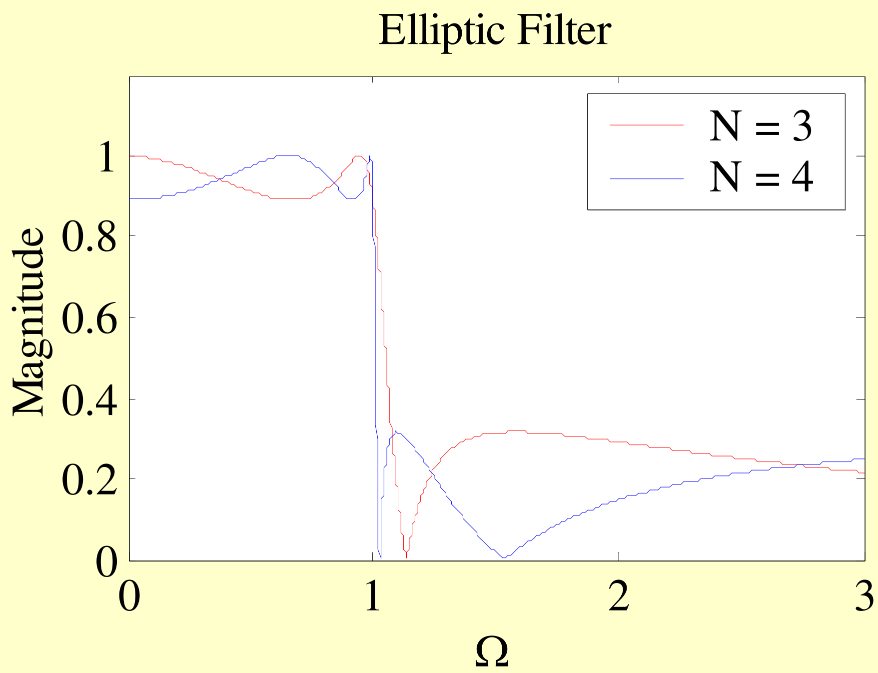
where $k' = \sqrt{1-k^2}$

$$\rho_0 = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}$$

$$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13}$$

Elliptic Approximation

- Typical magnitude response plots with $\Omega_p = 1$ are shown below



Design of Analog Highpass, Bandpass and Bandstop Filters

- Steps involved in the design process:

Step 1 - Develop of specifications of a prototype analog lowpass filter $H_{LP}(s)$ from specifications of desired analog filter $H_D(s)$ using a frequency transformation

Step 2 - Design the prototype analog lowpass filter

Step 3 - Determine the transfer function $H_D(s)$ of desired analog filter by applying the inverse frequency transformation to $H_{LP}(s)$

Design of Analog Highpass, Bandpass and Bandstop Filters

- Let s denote the Laplace transform variable of prototype analog lowpass filter $H_{LP}(s)$ and $\hat{s}$ denote the Laplace transform variable of desired analog filter $H_D(\hat{s})$
- The mapping from s-domain to $\hat{s}$-domain is given by the invertible transformation

$$s = F(\hat{s})$$

- Then $H_D(\hat{s}) = H_{LP}(s)\big|_{s=F(\hat{s})}$

$$H_{LP}(s) = H_D(\hat{s})\big|_{\hat{s}=F^{-1}(s)}$$

29

Analog Highpass Filter Design

- Spectral Transformation:

$$s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

normalized version:
$s = 1/\hat{s}$

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$

- On the imaginary axis the transformation is

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

Analog Bandpass Filter Design

normalized version:
$s = (\hat{s}^2 + 1)/\hat{s}$

- Spectral Transformation

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ are the lower and upper passband edge frequencies of desired bandpass filter $H_{BP}(\hat{s})$

Analog Bandpass Filter Design

- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies
- Also,

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$$

Geometric symmetry of the passband

- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied

Analog Bandstop Filter Design

normalized version:

$s = \hat{s}/(\hat{s}^2 + 1)$

- Spectral Transformation

$$s = \Omega_s \frac{\hat{s}(\hat{\Omega}_{s2} - \hat{\Omega}_{s1})}{\hat{s}^2 + \hat{\Omega}_o^2}$$

where Ω_s is the stopband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{s1}$ and $\hat{\Omega}_{s2}$ are the lower and upper stopband edge frequencies of the desired bandstop filter $H_{BS}(\hat{s})$

Analog Bandstop Filter Design

- On the imaginary axis the transformation is

$$\Omega = \Omega_s \frac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$$

  where $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1}$ is the width of stopband and $\hat{\Omega}_o$ is the **stopband center frequency** of the bandstop filter

- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies