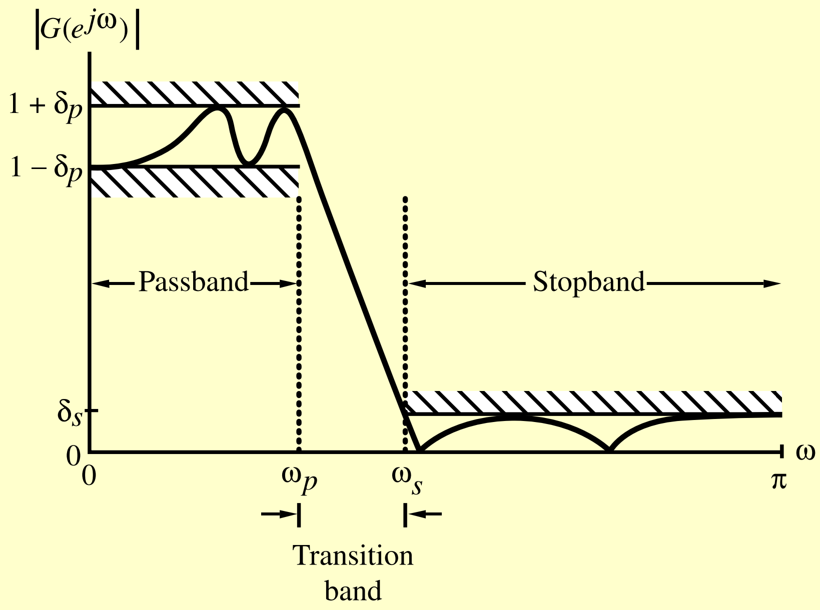
# Digital Filter Design

- Objective - Determination of a realizable transfer function $G(z)$ approximating a given frequency response specification is an important step in the development of a digital filter
- If an IIR filter is desired, $G(z)$ should be a stable real rational function
- Digital filter design is the process of deriving the transfer function $G(z)$

# Digital Filter Specifications

- For example, the magnitude response $|G(e^{j\omega})|$ of a digital lowpass filter with a real rational transfer function $G(z)$ may be given as indicated below



# Digital Filter Specifications

- As indicated in the figure, in the **passband**, defined by $0 \le \omega \le \omega_p$, we require that $\left|G(e^{j\omega})\right| \cong 1$ with an error $\pm\delta_p$, i.e.,

$$1-\delta_p \le \left|G(e^{j\omega})\right| \le 1+\delta_p, \quad |\omega| \le \omega_p$$

- In the **stopband**, defined by $\omega_s \le \omega \le \pi$, we require that $\left|G(e^{j\omega})\right| \cong 0$ with an error $\delta_s$, i.e.,

$$\left|G(e^{j\omega})\right| \le \delta_s, \quad \omega_s \le |\omega| \le \pi$$

# Digital Filter Specifications

- $\omega_p$ - **passband edge frequency**
- $\omega_s$ - **stopband edge frequency**
- $\delta_p$ - **peak ripple value** in the passband
- $\delta_s$ - **peak ripple value** in the stopband
- Since $G(e^{j\omega})$ is a periodic function of $\omega$, and $|G(e^{j\omega})|$ of a real-coefficient digital filter is an even function of $\omega$
- As a result, filter specifications are given only for the frequency range $0 \leq \omega \leq \pi$

# Digital Filter Specifications

- Specifications are often given in terms of **loss function** $\mathcal{A}(\omega) = -20\log_{10}\left|G(e^{j\omega})\right|$ in dB
- **Peak passband ripple**

$$\alpha_p = -20\log_{10}(1-\delta_p) \text{ dB}$$

- **Minimum stopband attenuation**

$$\alpha_s = -20\log_{10}(\delta_s) \text{ dB}$$

# Digital Filter Specifications

- In practice, passband edge frequency $F_p$ and stopband edge frequency $F_s$ may be specified in Hz (or other analog units, say cycle/cm)
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

# Selection of Filter Type

- The transfer function $H(z)$ meeting the frequency response specifications should be a causal transfer function
- For IIR digital filter design, the IIR transfer function is a real rational function of $z^{-1}$:

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}, \quad M \le N$$

- $H(z)$ must be a stable transfer function and must be of lowest order $N$ for reduced computational complexity

# Selection of Filter Type

- For FIR digital filter design, the FIR transfer function is a polynomial in $z^{-1}$ with real coefficients:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

- For reduced computational complexity, degree $N$ of $H(z)$ must be as small as possible
- If a linear phase is desired, the filter coefficients must satisfy the constraint:

$$h[n] = \pm h[N-n]$$

# Selection of Filter Type

- Advantages in using an FIR filter -

  (1) Can be designed with exact linear phase,

  (2) Filter structure always stable with quantized coefficients

- Disadvantages in using an FIR filter - Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

# Digital Filter Design: Basic Approaches

- Most common approach to IIR filter design -
  (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications
- (2) Determine the analog lowpass filter transfer function $H_a(s)$ see Appendix, later
- (3) Transform $H_a(s)$ into the desired digital transfer function $G(z)$

# Digital Filter Design: Basic Approaches

- FIR filter design is based on a direct approximation of the specified magnitude response, with the often added requirement that the phase be linear
- The design of an FIR filter of order $N$ may be accomplished by finding either the length-($N$+1) impulse response samples $\{h[n]\}$ or the ($N$+1) samples of its frequency response $H(e^{j\omega})$

# Digital Filter Design: Basic Approaches

- Three commonly used approaches to FIR filter design -

  (1) Windowed Fourier series approach

  (2) Frequency sampling approach

  (3) Computer-based optimization methods

see Appendix now

# IIR Digital Filter Design: Bilinear Transformation Method

- Bilinear transformation -

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

- Above transformation maps a single point in the $s$-plane to a unique point in the $z$-plane and vice-versa
- Relation between $G(z)$ and $H_a(s)$ is then given by

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

# Bilinear Transformation

- Digital filter design consists of 3 steps:

  (1) Develop the specifications of $H_a(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$

  (2) Design $H_a(s)$

  (3) Determine $G(z)$ by applying bilinear transformation to $H_a(s)$

- As a result, the parameter $T$ has no effect on $G(z)$ and $T = 2$ is chosen for convenience

# Bilinear Transformation

- Inverse bilinear transformation for $T = 2$ is

$$z = \frac{1+s}{1-s}$$

- For $s = \sigma_o + j\Omega_o$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \quad \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

- Thus,

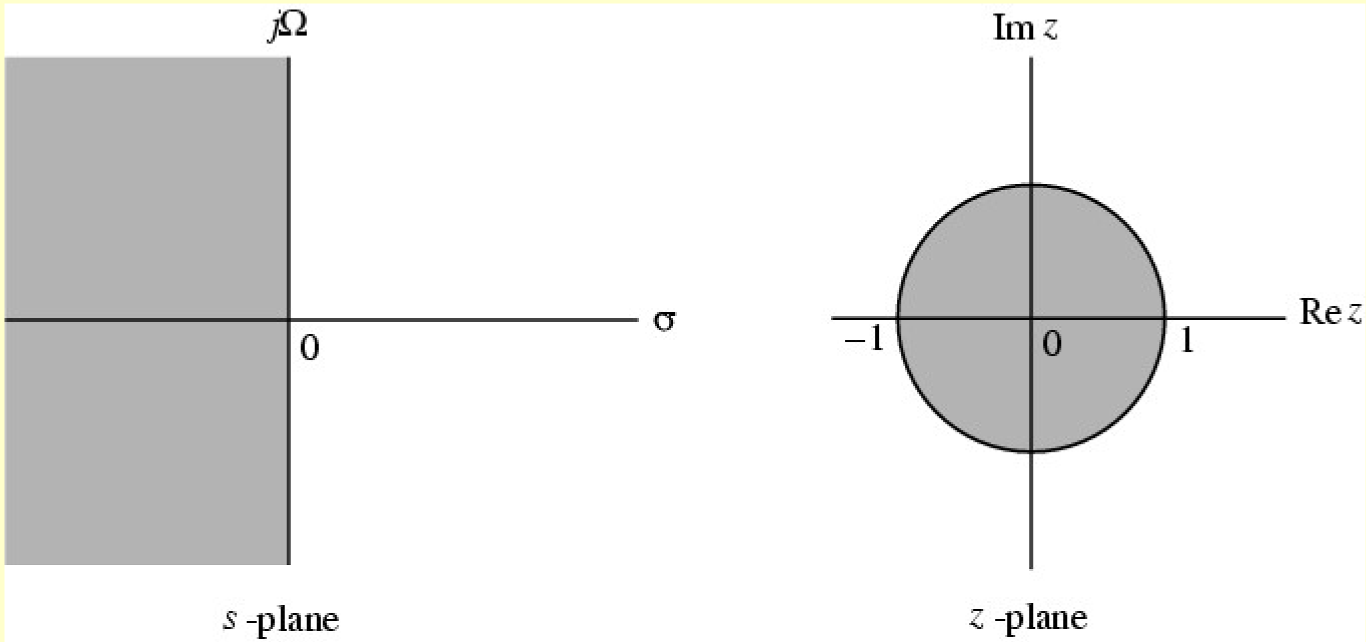
$$\sigma_o = 0 \rightarrow |z| = 1$$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

# Bilinear Transformation

- Mapping of *s*-plane into the *z*-plane



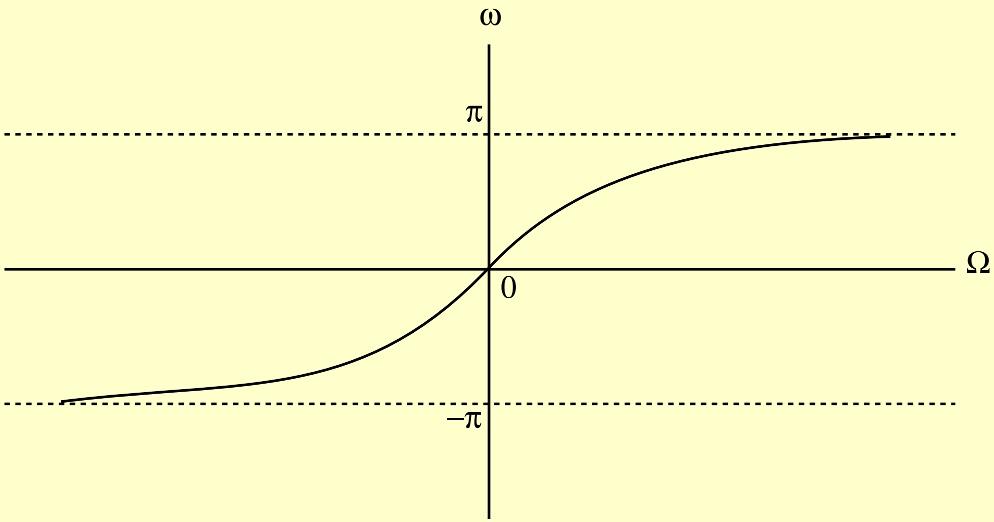
# Bilinear Transformation

- For $z = e^{j\omega}$ with $T = 2$ we have

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2})}$$

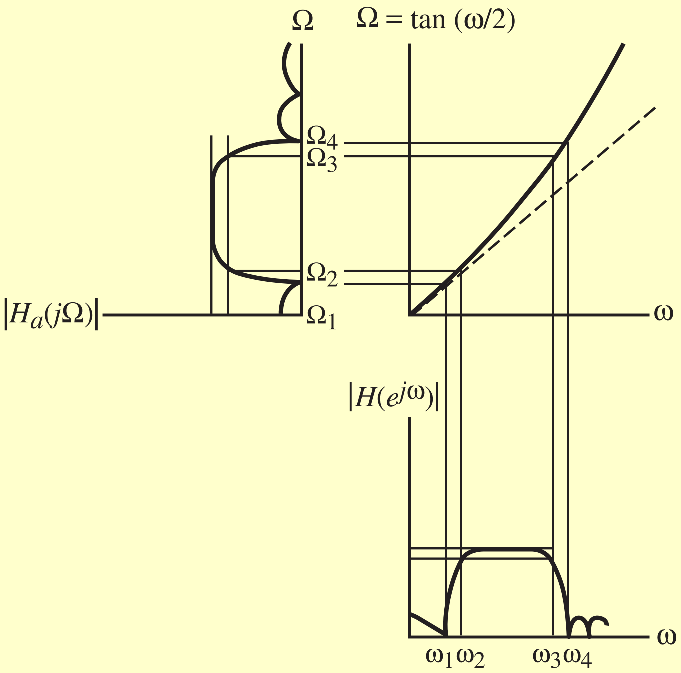
$$= \frac{j2\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2)$$

or $\Omega = \tan(\omega/2)$



# Bilinear Transformation

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- Effect of warping shown below





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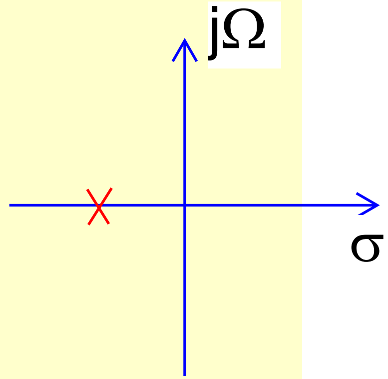
# IIR Digital Filter Design Using Bilinear Transformation




- Example - Consider

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

- Applying bilinear transformation to the above we get the transfer function of a first-order digital lowpass Butterworth filter

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1})+\Omega_c(1+z^{-1})}$$

# IIR Digital Filter Design Using Bilinear Transformation

- Rearranging terms we get

$$G(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha\, z^{-1}}$$

where

$$\alpha = \frac{1-\Omega_c}{1+\Omega_c} = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}$$

same filter as in slide 4.6.21,
same $\alpha$ (even if it looks different)

# Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

- First Approach -

  (1) Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type

  (2) Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$

  (3) Design analog lowpass filter $H_{LP}(s)$

# Design of IIR Highpass, Bandpass, and Bandstop Digital Filters

(4) Convert $H_{LP}(s)$ into $H_D(s)$ using inverse frequency transformation used in Step 2

(5) Design desired digital filter $G_D(z)$ by applying bilinear transformation to $H_D(s)$

# IIR Highpass Digital Filter Design

- Design of a Type 1 Chebyshev IIR digital highpass filter
- Specifications: $F_p = 700$ Hz, $F_s = 500$ Hz, $\alpha_p = 1$ dB, $\alpha_s = 32$ dB, $F_T = 2$ kHz
- Normalized angular bandedge frequencies

$$\omega_p = \frac{2\pi F_p}{F_T} = \frac{2\pi \times 700}{2000} = 0.7\pi$$

$$\omega_s = \frac{2\pi F_s}{F_T} = \frac{2\pi \times 500}{2000} = 0.5\pi$$

# IIR Highpass Digital Filter Design

- Prewarping these frequencies we get

$$\hat{\Omega}_p = \tan(\omega_p/2) = 1.9626105$$

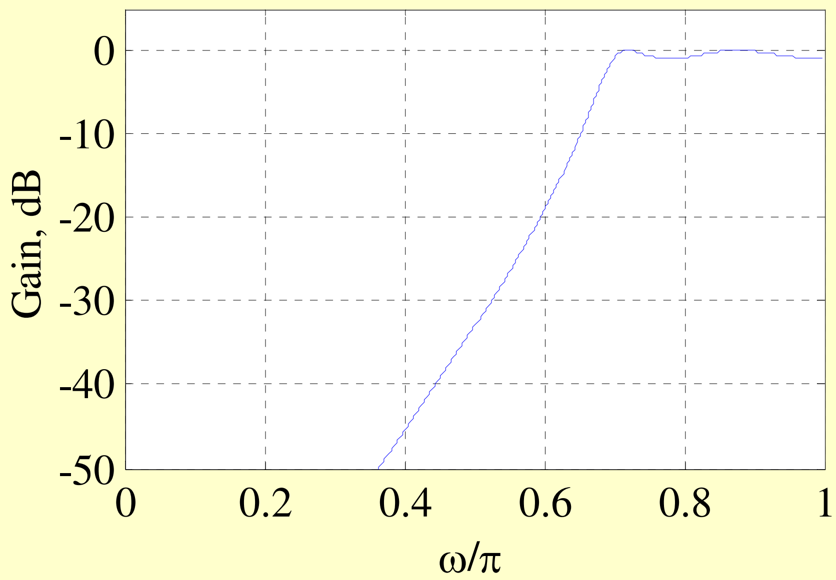
$$\hat{\Omega}_s = \tan(\omega_s/2) = 1.0$$

- For the prototype analog lowpass filter choose $\Omega_p = 1$
- Using $\Omega = -\dfrac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$ we get $\Omega_s = 1.9626105$
- Analog lowpass filter specifications: $\Omega_p = 1$, $\Omega_s = 1.9626105$, $\alpha_p = 1$dB, $\alpha_s = 32$dB

# IIR Highpass Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = cheb1ord(1, 1.9626105, 1, 32, 's')
[B, A] = cheby1(N, 1, Wn, 's');
[BT, AT] = lp2hp(B, A, 1.9626105);
[num, den] = bilinear(BT, AT, 0.5);
```



# Computer-Aided Design of IIR Digital Filter

- The IIR filter design algorithms discussed so far are used in applications requiring filters with a frequency-selective magnitude response having either a lowpass or a highpass or a bandpass or a bandstop characteristics with **piecewise uniform** shape
- Designing IIR filters with other types of frequency responses usually involve the use of some type of iterative optimization techniques that are used to minimize the error between the desired response and that of the computer-generated filter

# Computer-Aided Design of IIR Digital Filter

- **Basic Idea -**
- Let $G(e^{j\omega})$ denote the frequency response of the computer generated transfer function $G(z)$
- Let $D(e^{j\omega})$ denote the desired frequency response
- Objective is to design $G(z)$ so that $G(e^{j\omega})$ approximates $D(e^{j\omega})$ in some sense

# IIR Digital Filter Design Using Optimization Techniques

- In the case of IIR digital filter design, $G(e^{j\omega})$ and $D(e^{j\omega})$ are replaced with their magnitude functions
- Consider the real rational transfer function

$$G(z) = C\frac{1+p_1 z^{-1}+\ldots+p_M z^{-M}}{1+d_1 z^{-1}+\ldots+d_N z^{-N}}$$

where the coefficients $p_\ell$ and $d_\ell$ are real, and the gain constant $C$ is a positive number

# IIR Digital Filter Design Using Optimization Techniques

- Let $|G(\mathbf{x},\omega)|$ denote the magnitude response of $G(z)$ where $\mathbf{x}$ represents the column vector of the adjustable parameters consisting of the filter coefficients $p_\ell$ and $d_\ell$, the gain constant $C$; that is,

$$\mathbf{x} = [p_1,\ p_2,\ \ldots,\ p_M,\ d_1,\ d_2,\ \ldots,\ d_N,\ C]$$

# IIR Digital Filter Design Using Optimization Techniques

- The approximation error at a given frequency $\omega$ is given by the weighted difference between $|G(\mathbf{x},\omega)|$ and the desired magnitude response $|D(e^{j\omega})|$:

$$\varepsilon(\mathbf{x},\omega) = W(\omega)(|G(\mathbf{x},\omega)| - |D(\omega)|)$$

with $W(\omega)$ denoting the user specified weighting function

# IIR Digital Filter Design Using Optimization Techniques

- The error vector is obtained by evaluating the error $\varepsilon(\mathbf{x},\omega)$ at a dense grid of frequency points, $\omega_1, \omega_2, \ldots, \omega_K$:

$$\mathbf{E}(\mathbf{x}) = [\varepsilon(\mathbf{x},\omega_1), \varepsilon(\mathbf{x},\omega_2), \ldots, \varepsilon(\mathbf{x},\omega_K)]$$

- The filter design method is based on the adjustment of the parameters of $\mathbf{x}$ iteratively until a set of values $\mathbf{x} = \hat{\mathbf{x}}$ is found for which $\varepsilon(\hat{\mathbf{x}},\omega_i) \approx 0$ for $i = 0, 1, \ldots, K$

# IIR Digital Filter Design Using Optimization Techniques

- A commonly used approximation measure is the least-$p$th objective function given by

$$\varepsilon(\mathbf{x}) = \hat{\mathbf{E}}(\mathbf{x})\left\{\sum_{i=1}^{K}\left[\frac{\varepsilon(\mathbf{x},\omega_i)}{\hat{\mathbf{E}}(\mathbf{x})}\right]^p\right\}^{1/p}$$

where

$$\hat{\mathbf{E}}(\mathbf{x}) = \max_{1\le i\le K}\left|\varepsilon(\mathbf{x},\omega_i)\right|$$

# IIR Digital Filter Design Using Optimization Techniques

- The MATLAB function `iirlpnorm` employs am uncostrained quasi-Newton algorithm to minimize the objective function $\varepsilon(\mathbf{x})$
- If at any stage of the iteration process one or more poles and/or zeros of $G(z)$ lie outside the unit circle, they are reflected back to inside the unit circle, which does not change the magnitude function $|G(e^{j\omega})|$



Matlab: filterDesigner