Simple Digital Filters

- Later in the course we shall review various methods of designing frequency-selective filters satisfying prescribed specifications
- We now describe several low-order FIR and IIR digital filters with reasonable selective frequency responses that often are satisfactory in a number of applications

Simple FIR Digital Filters

- FIR digital filters considered here have integer-valued impulse response coefficients
- These filters are employed in a number of practical applications, primarily because of their simplicity, which makes them amenable to inexpensive hardware implementations

Simple FIR Digital Filters

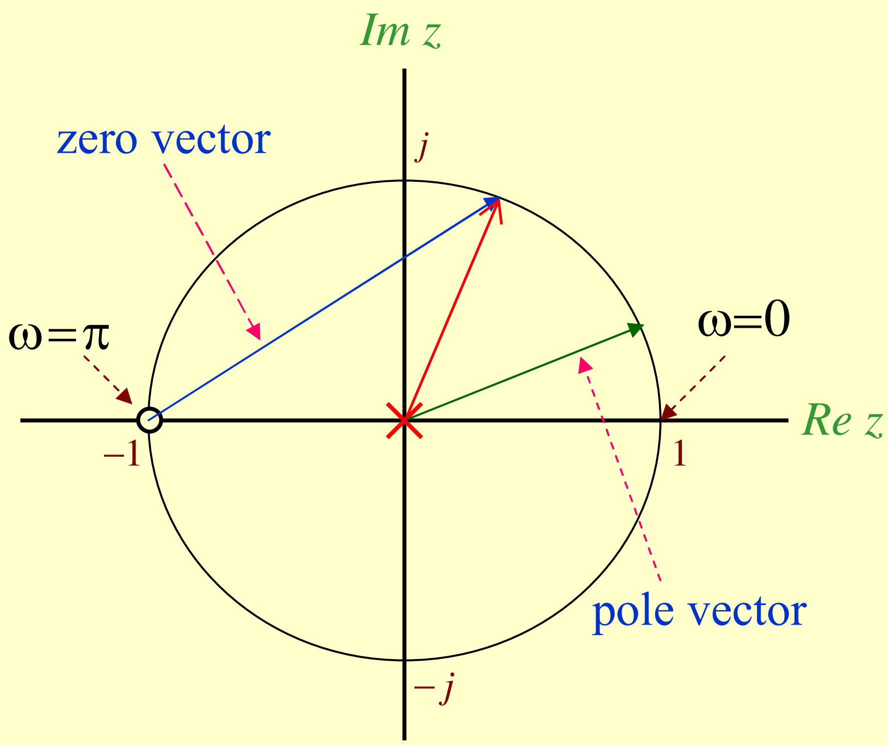
Lowpass FIR Digital Filters

- The simplest lowpass FIR digital filter is the 2-point moving-average filter given by

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1}) = \frac{z+1}{2z}$$

- The above transfer function has a zero at $z = -1$ and a pole at $z = 0$
- Note that here the pole vector has a unity magnitude for all values of ω

Simple FIR Digital Filters



Simple FIR Digital Filters

- On the other hand, as ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2, the diameter of the unit circle, to 0
- Hence, the magnitude response $|H_0(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$

Simple FIR Digital Filters

- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$, i.e.,

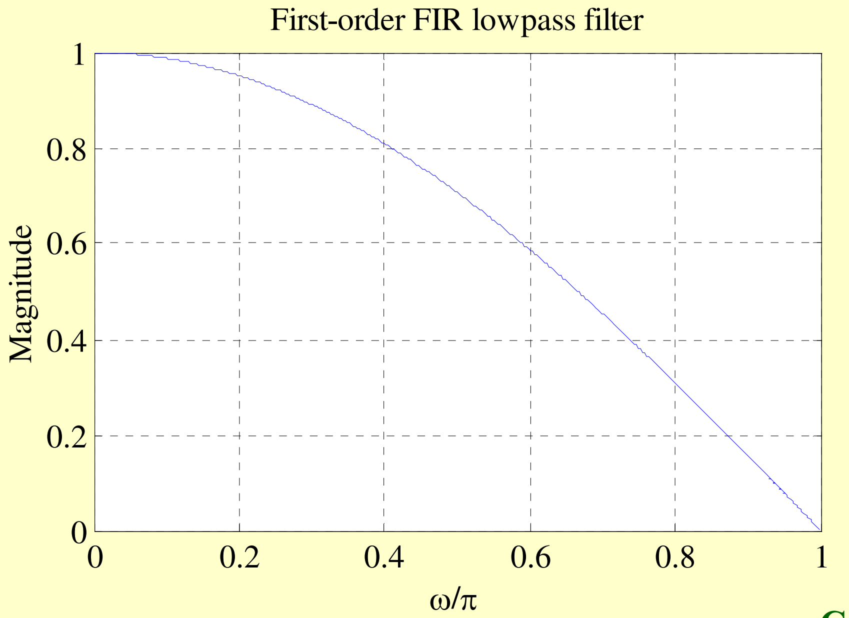
$$|H_0(e^{j0})| = 1, \quad |H_0(e^{j\pi})| = 0$$

- The frequency response of the above filter is given by

$$H_0(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2)$$

Simple FIR Digital Filters

- The magnitude response $|H_0(e^{j\omega})| = \cos(\omega/2)$ can be seen to be a monotonically decreasing function of ω



Simple FIR Digital Filters

- The frequency $\omega = \omega_c$ at which

$$\left|H_0(e^{j\omega_c})\right| = \frac{1}{\sqrt{2}}\left|H_0(e^{j0})\right|$$

is of practical interest since here the gain $\mathcal{G}(\omega_c)$ in dB is given by

$$\mathcal{G}(\omega_c) = 20\log_{10}\left|H(e^{j\omega_c})\right|$$
$$= 20\log_{10}\left|H(e^{j0})\right| - 20\log_{10}\sqrt{2} \cong -3 \text{ dB}$$

since the dc gain $\mathcal{G}(0) = 20\log_{10}\left|H(e^{j0})\right| = 0$

Simple FIR Digital Filters

- Thus, the gain $\mathcal{G}(\omega)$ at $\omega = \omega_c$ is approximately 3 dB less than the gain at $\omega = 0$
- As a result, ω_c is called the **3-dB cutoff frequency**
- To determine the value of ω_c we set

$$|H_0(e^{j\omega_c})|^2 = \cos^2(\omega_c/2) = \frac{1}{2}$$

which yields $\omega_c = \pi/2$

Simple FIR Digital Filters

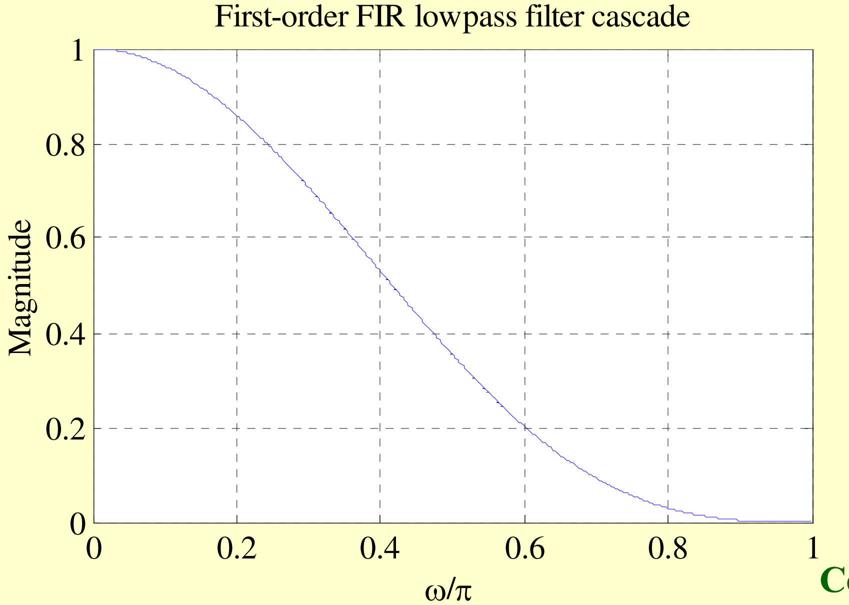
- The 3-dB cutoff frequency ω_c can be considered as the passband edge frequency
- As a result, for the filter $H_0(z)$ the passband width is approximately $\pi/2$
- The stopband is from $\pi/2$ to π
- Note: $H_0(z)$ has a zero at $z = -1$ or $\omega = \pi$, which is in the stopband of the filter

Simple FIR Digital Filters

- A cascade of the simple FIR filter

$$H_0(z) = \tfrac{1}{2}(1 + z^{-1})$$

results in an improved lowpass frequency response as illustrated below for a cascade of 3 sections



What are the coeffs. of the impulse response of this filter ?

11

Simple FIR Digital Filters

- The 3-dB cutoff frequency of a cascade of M sections is given by

$$\omega_c = 2\cos^{-1}(2^{-1/2M})$$

- For $M = 3$, the above yields $\omega_c = 0.302\pi$
- Thus, the cascade of first-order sections yields a sharper magnitude response but at the expense of a decrease in the width of the passband

Simple FIR Digital Filters

- A better approximation to the ideal lowpass filter is given by a higher-order moving-average filter

Compare the two filters using Matlab

- Signals with rapid fluctuations in sample values are generally associated with high-frequency components
- These high-frequency components are essentially removed by an moving-average filter resulting in a smoother output waveform

Simple FIR Digital Filters

Highpass FIR Digital Filters

- The simplest highpass FIR filter is obtained from the simplest lowpass FIR filter by replacing z with $-z$

i.e.: $\exp(j\omega)$ ⇨ $-\exp(j\omega) = \exp(j(\omega+\pi))$

- This results in

$$H_1(z) = \frac{1}{2}(1 - z^{-1})$$

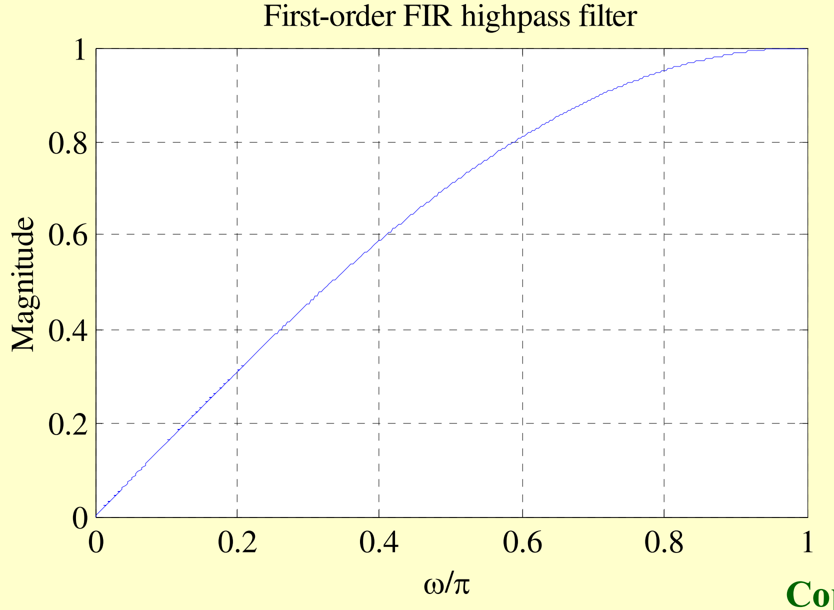
First derivative filter (backward)

Simple FIR Digital Filters

- Corresponding frequency response is given by

$$H_1(e^{j\omega}) = j\, e^{-j\omega/2} \sin(\omega/2)$$

whose magnitude response is plotted below



Simple FIR Digital Filters

- The monotonically increasing behavior of the magnitude function can again be demonstrated by examining the pole-zero pattern of the transfer function $H_1(z)$
- The highpass transfer function $H_1(z)$ has a zero at $z = 1$ or $\omega = 0$ which is in the stopband of the filter

16

Simple FIR Digital Filters

- Improved highpass magnitude response can again be obtained by cascading several sections of the first-order highpass filter
- Alternately, a higher-order highpass filter of the form

$$H_1(z) = \frac{1}{M}\sum_{n=0}^{M-1}(-1)^n z^{-n}$$

is obtained by replacing z with $-z$ in the transfer function of a moving average filter

Simple FIR Digital Filters

- An application of the FIR highpass filters is in moving-target-indicator (MTI) radars
- In these radars, interfering signals, called **clutters**, are generated from fixed objects in the path of the radar beam
- The clutter, generated mainly from ground echoes and weather returns, has frequency components near zero frequency (dc)

18

Simple FIR Digital Filters

- The clutter can be removed by filtering the radar return signal through a **two-pulse canceler**, which is the first-order FIR highpass filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$
- For a more effective removal it may be necessary to use a **three-pulse canceler** obtained by cascading two two-pulse cancelers

I.e.: derivative of the derivative
--> second derivative filter

19

Simple IIR Digital Filters

Lowpass IIR Digital Filters

- We have shown earlier that the first-order causal IIR transfer function

$$H(z) = \frac{K}{1-\alpha z^{-1}}, \quad 0 < \alpha < 1$$

has a lowpass magnitude response for $\alpha > 0$

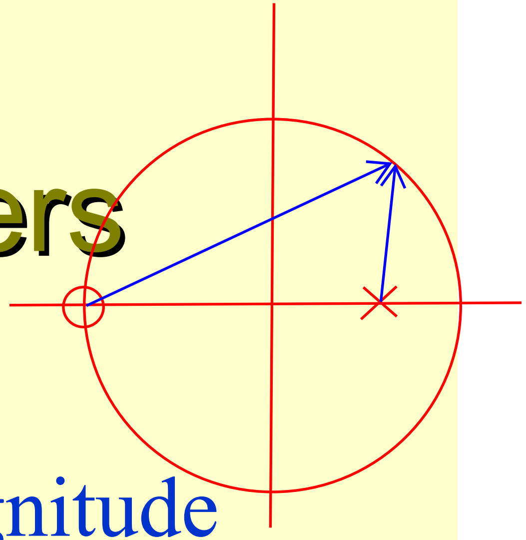
Simple IIR Digital Filters

- An improved lowpass magnitude response is obtained by adding a factor $(1+z^{-1})$ to the numerator of transfer function

$$H(z) = \frac{K(1+z^{-1})}{1-\alpha z^{-1}}, \quad 0<\alpha<1$$

- This forces the magnitude response to have a zero at $\omega = \pi$ in the stopband of the filter

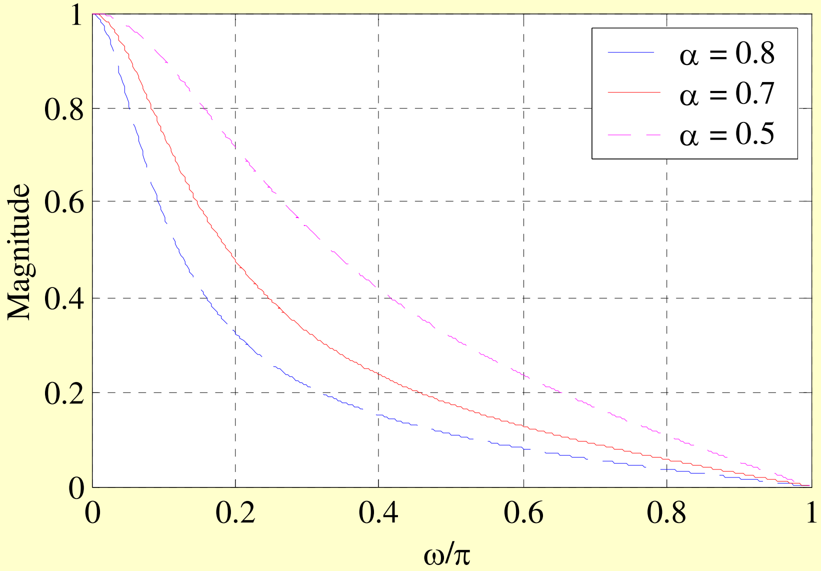
Simple IIR Digital Filters

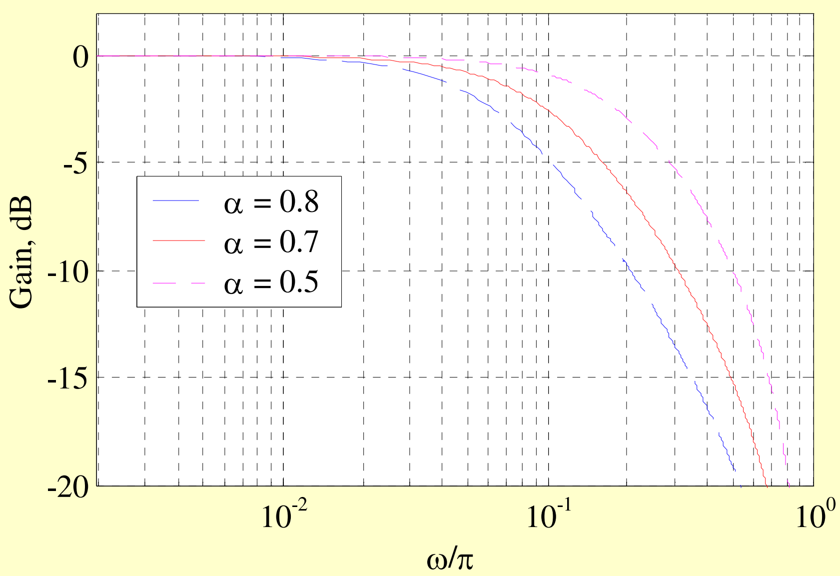


- $H_{LP}(z)$ has a real pole at $z = \alpha$
- As ω increases from 0 to π, the magnitude of the zero vector decreases from a value of 2 to 0, whereas, for a positive value of α, the magnitude of the pole vector increases from a value of $1-\alpha$ to $1+\alpha$
- The maximum value of the magnitude function is 1 at $\omega = 0$, and the minimum value is 0 at $\omega = \pi$

Simple IIR Digital Filters

- i.e., $|H_{LP}(e^{j0})| = 1, \quad |H_{LP}(e^{j\pi})| = 0$
- Therefore, $|H_{LP}(e^{j\omega})|$ is a monotonically decreasing function of ω from $\omega = 0$ to $\omega = \pi$ as indicated below





Simple IIR Digital Filters

- The squared magnitude function is given by

$$|H_{LP}(e^{j\omega})|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

- The derivative of $|H_{LP}(e^{j\omega})|^2$ with respect to ω is given by

$$\frac{d\,|H_{LP}(e^{j\omega})|^2}{d\omega} = \frac{-(1-\alpha)^2(1+2\alpha+\alpha^2)\sin\omega}{2(1-2\alpha\cos\omega+\alpha^2)^2}$$

Simple IIR Digital Filters

$d\,|H_{LP}(e^{j\omega})|^2/d\omega \leq 0$ in the range $0 \leq \omega \leq \pi$ verifying again the monotonically decreasing behavior of the magnitude function

- To determine the 3-dB cutoff frequency we set

$$|H_{LP}(e^{j\omega_c})|^2 = \frac{1}{2}$$

in the expression for the square magnitude function resulting in

Simple IIR Digital Filters

$$\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}=\frac{1}{2}$$

or

$$(1-\alpha)^2(1+\cos\omega_c)=1+\alpha^2-2\alpha\cos\omega_c$$

which when solved yields

$$\cos\omega_c=\frac{2\alpha}{1+\alpha^2}$$

- The above quadratic equation can be solved for α yielding two solutions

Simple IIR Digital Filters

- The solution resulting in a stable transfer function $H_{LP}(z)$ is given by

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

Simple IIR Digital Filters

Highpass IIR Digital Filters

- A first-order causal highpass IIR digital filter has a transfer function given by

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

from the LP filter:
$z \Rightarrow -z$ and
redefine α

where $|\alpha| < 1$ for stability

- The above transfer function has a zero at $z = 1$ i.e., at $\omega = 0$ which is in the stopband

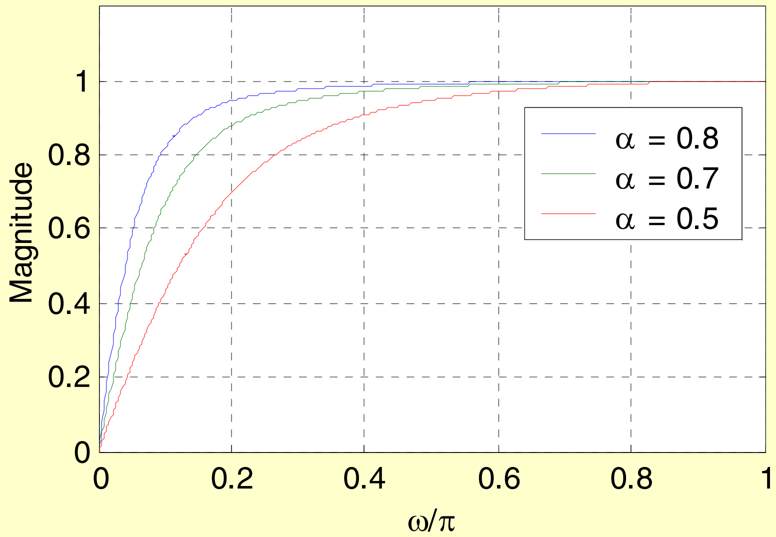
Simple IIR Digital Filters

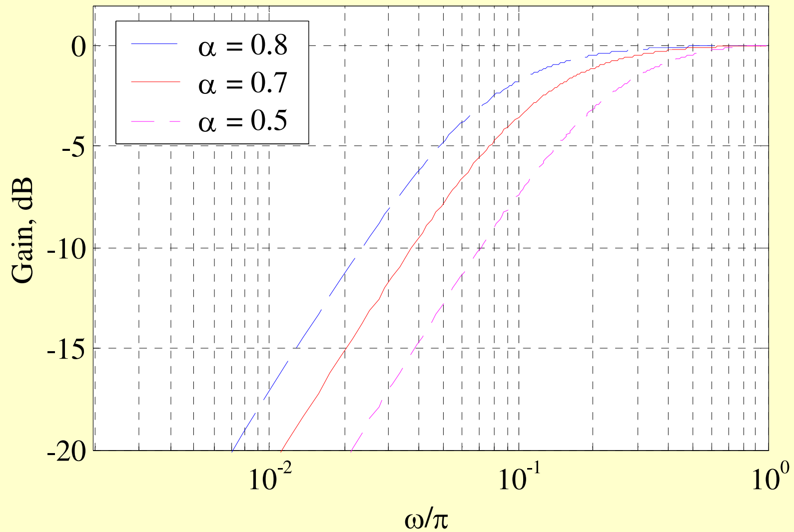
- Its 3-dB cutoff frequency ω_c is given by

$$\alpha = (1 - \sin\omega_c)/\cos\omega_c$$

which is the same as that of $H_{LP}(z)$

- Magnitude and gain responses of $H_{HP}(z)$ are shown below





Simple IIR Digital Filters

- $H_{HP}(z)$ is a BR function for $|\alpha| < 1$
- Example - Design a first-order highpass digital filter with a 3-dB cutoff frequency of 0.8π
- Now, $\sin(\omega_c) = \sin(0.8\pi) = 0.587785$ and $\cos(0.8\pi) = -0.80902$
- Therefore
$$\alpha = (1 - \sin\omega_c)/\cos\omega_c = -0.5095245$$

Simple IIR Digital Filters

- Therefore,

$$H_{HP}(z) = \frac{1+\alpha}{2}\left(\frac{1-z^{-1}}{1-\alpha z^{-1}}\right)$$

$$= 0.245238\left(\frac{1-z^{-1}}{1+0.5095245\, z^{-1}}\right)$$

Simple IIR Digital Filters

Bandpass IIR Digital Filters

can be also expressed as a LP+HP cascade

- A 2nd-order bandpass digital transfer function is given by

$$H_{BP}(z) = \frac{1-\alpha}{2}\left(\frac{1-z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- Its squared magnitude function is

$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{(1-\alpha)^2(1-\cos 2\omega)}{2[1+\beta^2(1+\alpha)^2+\alpha^2-2\beta(1+\alpha)^2\cos\omega+2\alpha\cos 2\omega]}$$

Simple IIR Digital Filters

- $|H_{BP}(e^{j\omega})|^2$ goes to zero at $\omega = 0$ and $\omega = \pi$
- It assumes a maximum value of 1 at $\omega = \omega_o$, called the **center frequency** of the bandpass filter, where

$$\omega_o = \cos^{-1}(\beta)$$

- The frequencies ω_{c1} and ω_{c2} where $|H_{BP}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**

Simple IIR Digital Filters

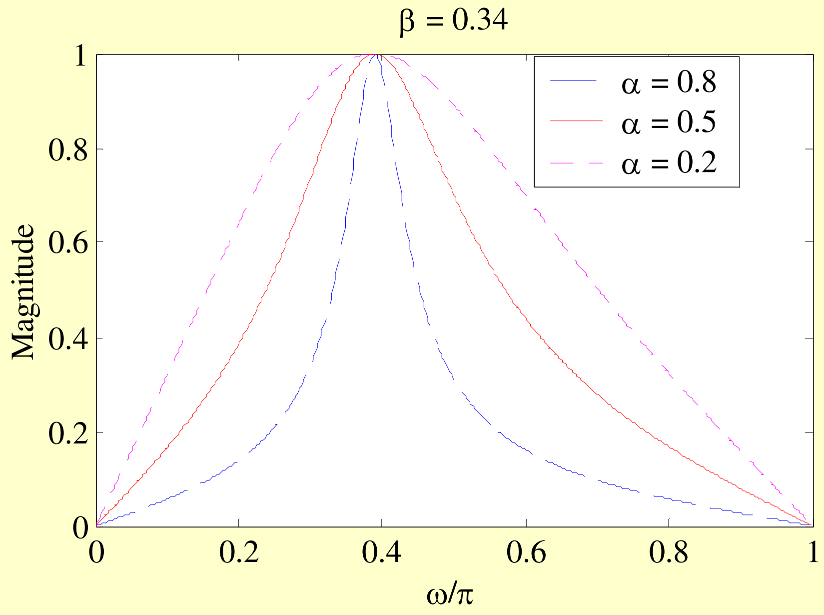
- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB bandwidth** and is given by

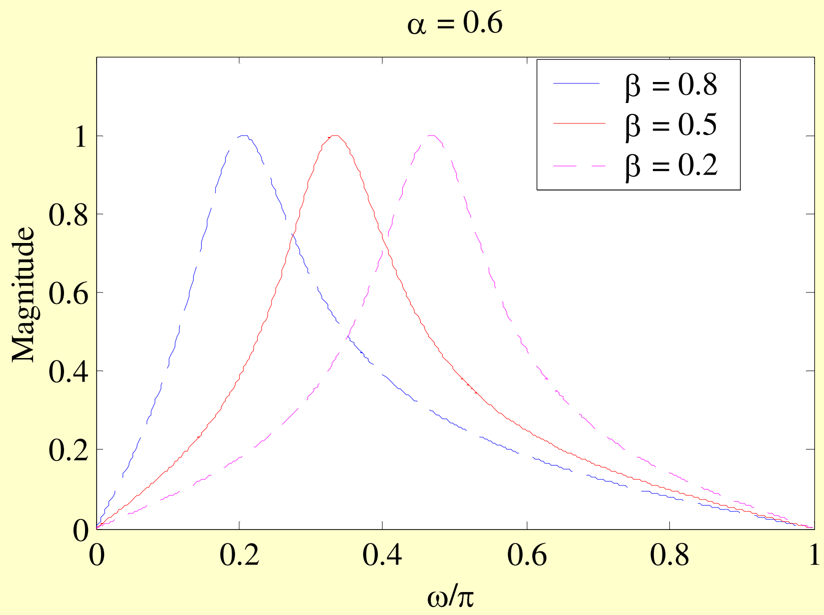
$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

- The transfer function $H_{BP}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

Simple IIR Digital Filters

- Plots of $|H_{BP}(e^{j\omega})|$ are shown below





Simple IIR Digital Filters

- Example - Design a 2nd order bandpass digital filter with center frequency at 0.4π and a 3-dB bandwidth of 0.1π
- Here $\beta = \cos(\omega_o) = \cos(0.4\pi) = 0.309017$

  and

$$\frac{2\alpha}{1+\alpha^2} = \cos(B_w) = \cos(0.1\pi) = 0.9510565$$

- The solution of the above equation yields: $\alpha = 1.376382$ and $\alpha = 0.72654253$

Simple IIR Digital Filters

- The corresponding transfer functions are

$$H'_{BP}(z) = -0.18819 \frac{1 - z^{-2}}{1 - 0.7343424 z^{-1} + 1.37638 z^{-2}}$$

and

$$H''_{BP}(z) = 0.13673 \frac{1 - z^{-2}}{1 - 0.533531 z^{-1} + 0.72654253 z^{-2}}$$

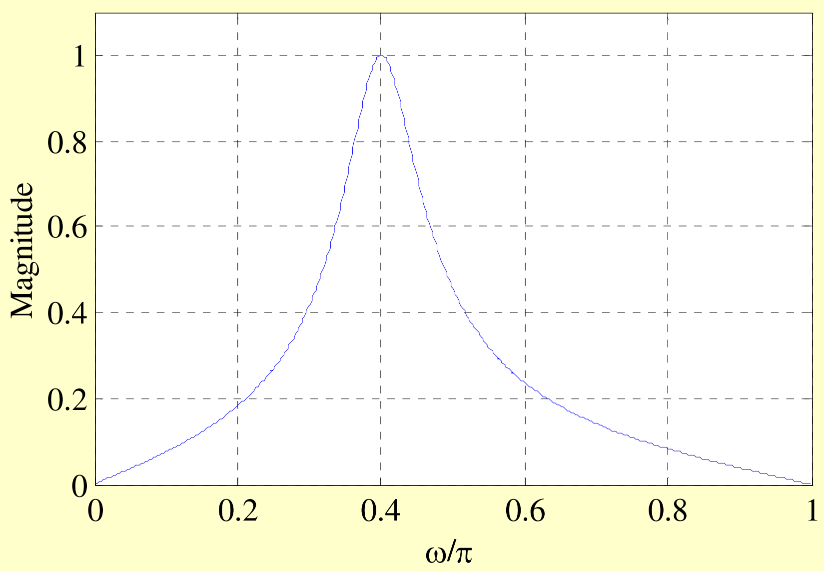
- The poles of $H'_{BP}(z)$ are at $z = 0.3671712 \pm j1.11425636$ and have a magnitude > 1

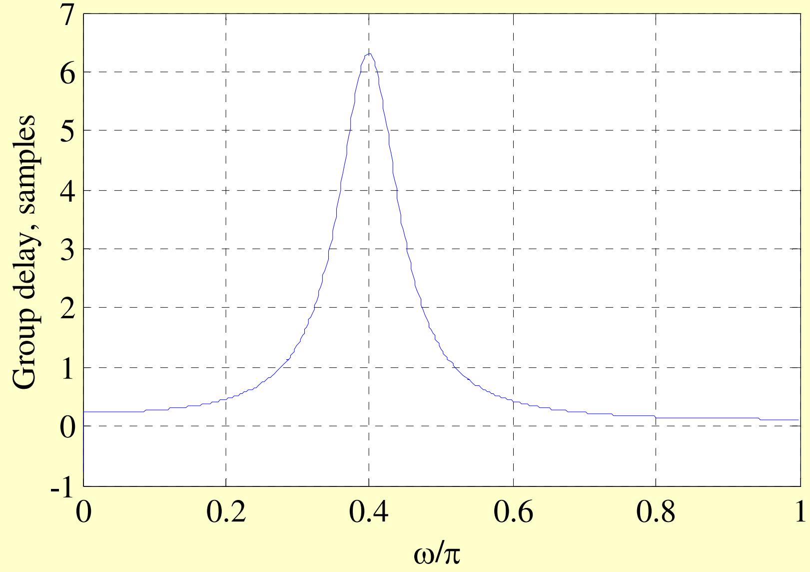
Simple IIR Digital Filters

- Thus, the poles of $H'_{BP}(z)$ are outside the unit circle making the transfer function unstable
- On the other hand, the poles of $H''_{BP}(z)$ are at $z = 0.2667655 \pm j0.8095546$ and have a magnitude of 0.8523746
- Hence $H''_{BP}(z)$ is BIBO stable
- Later we outline a simpler stability test

Simple IIR Digital Filters

- Figures below show the plots of the magnitude function and the group delay of $H''_{BP}(z)$





Simple IIR Digital Filters

Bandstop IIR Digital Filters

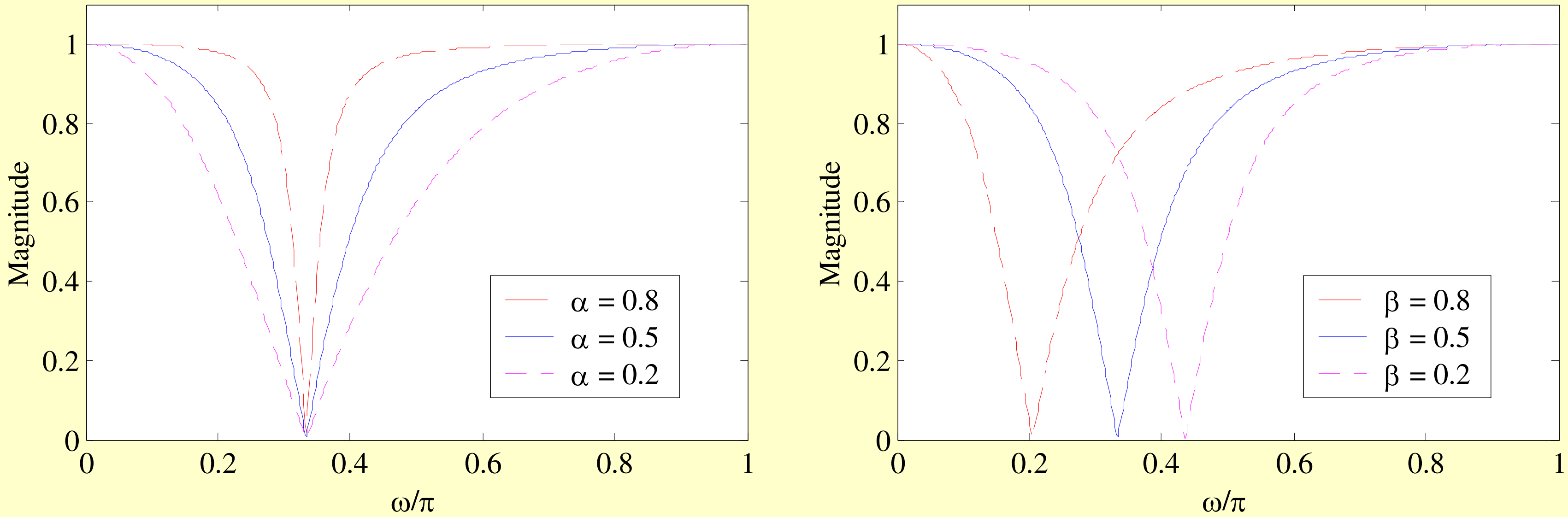
- A 2nd-order bandstop digital filter has a transfer function given by

$$H_{BS}(z) = \frac{1+\alpha}{2}\left(\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}\right)$$

- The transfer function $H_{BS}(z)$ is a BR function if $|\alpha| < 1$ and $|\beta| < 1$

Simple IIR Digital Filters

- Its magnitude response is plotted below



Simple IIR Digital Filters

- Here, the magnitude function takes the maximum value of 1 at $\omega = 0$ and $\omega = \pi$
- It goes to 0 at $\omega = \omega_o$, where ω_o , called the **notch frequency**, is given by

$$\omega_o = \cos^{-1}(\beta)$$

- The digital transfer function $H_{BS}(z)$ is more commonly called a **notch filter**

Simple IIR Digital Filters

- The frequencies ω_{c1} and ω_{c2} where $|H_{BS}(e^{j\omega})|^2$ becomes 1/2 are called the **3-dB cutoff frequencies**
- The difference between the two cutoff frequencies, assuming $\omega_{c2} > \omega_{c1}$ is called the **3-dB notch bandwidth** and is given by

$$B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

Simple IIR Digital Filters

Higher-Order IIR Digital Filters

- By cascading the simple digital filters discussed so far, we can implement digital filters with sharper magnitude responses
- Consider a cascade of K first-order lowpass sections characterized by the transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

Simple IIR Digital Filters

- The overall structure has a transfer function given by

$$G_{LP}(z) = \left(\frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}} \right)^K$$

- The corresponding squared-magnitude function is given by

$$|G_{LP}(e^{j\omega})|^2 = \left[\frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)} \right]^K$$

Simple IIR Digital Filters

- To determine the relation between its 3-dB cutoff frequency ω_c and the parameter α, we set

$$\left[\frac{(1-\alpha)^2(1+\cos\omega_c)}{2(1+\alpha^2-2\alpha\cos\omega_c)}\right]^K = \frac{1}{2}$$

which when solved for α, yields for a stable $G_{LP}(z)$:

$$\alpha = \frac{1+(1-C)\cos\omega_c - \sin\omega_c\sqrt{2C-C^2}}{1-C+\cos\omega_c}$$

Simple IIR Digital Filters

where

$$C = 2^{(K-1)/K}$$

- It should be noted that the expression for α given earlier reduces to

$$\alpha = \frac{1 - \sin\omega_c}{\cos\omega_c}$$

for $K = 1$

Simple IIR Digital Filters

- Example - Design a lowpass filter with a 3-dB cutoff frequency at $\omega_c = 0.4\pi$ using a single first-order section and a cascade of 4 first-order sections, and compare their gain responses
- For the single first-order lowpass filter we have

$$\alpha = \frac{1+\sin\omega_c}{\cos\omega_c} = \frac{1+\sin(0.4\pi)}{\cos(0.4\pi)} = 0.1584$$

Simple IIR Digital Filters

- For the cascade of 4 first-order sections, we substitute $K = 4$ and get

$$C = 2^{(K-1)/K} = 2^{(4-1)/4} = 1.6818$$

- Next we compute

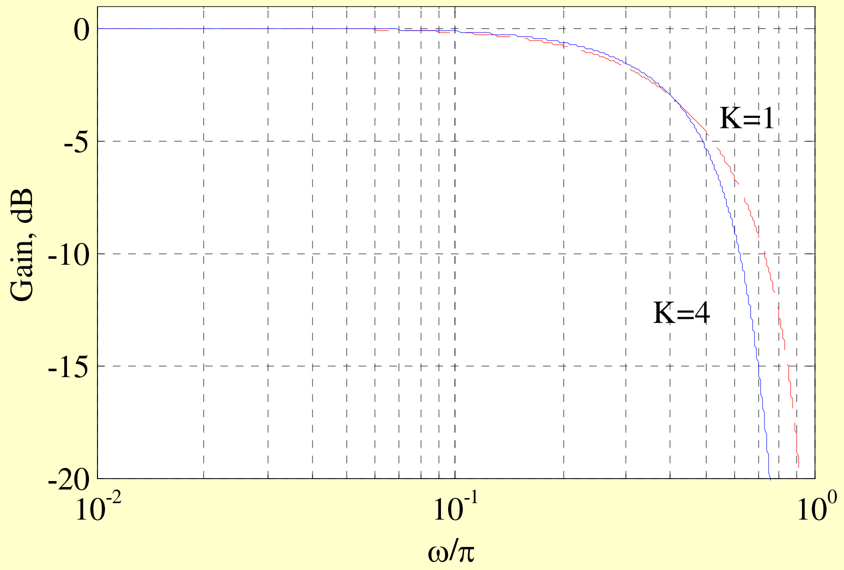
$$\alpha = \frac{1 + (1 - C)\cos\omega_c - \sin\omega_c\sqrt{2C - C^2}}{1 - C + \cos\omega_c}$$

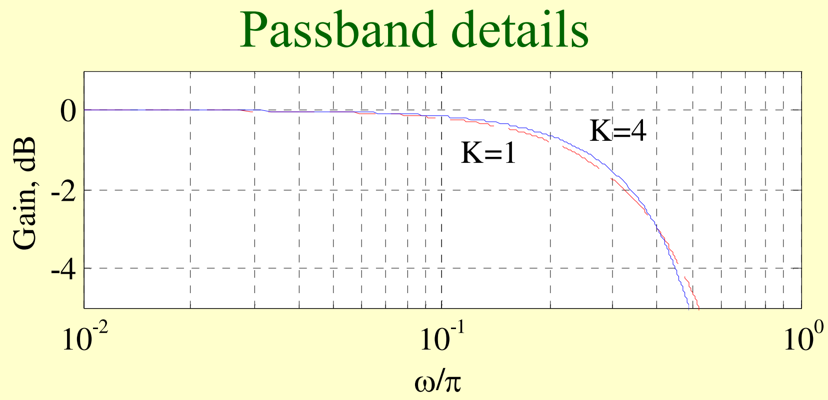
$$= \frac{1 + (1 - 1.6818)\cos(0.4\pi) - \sin(0.4\pi)\sqrt{2(1.6818) - (1.6818)^2}}{1 - 1.6818 + \cos(0.4\pi)}$$

$$= -0.251$$

Simple IIR Digital Filters

- The gain responses of the two filters are shown below
- As can be seen, cascading has resulted in a sharper roll-off in the gain response





Digital Differentiators

- Employed to perform the differentiation operation on the discrete-time version of a continuous-time signal
- Frequency response of an ideal discrete-time differentiator is given by

$$H(e^{j\omega}) = j\omega \quad \text{for } 0 \le |\omega \le \pi|$$

which has a linear magnitude response from dc to $\omega = \pi$

Digital Differentiators

- A practical discrete-time differentiator is used to perform the differentiation operation in the low frequency range and is thus designed to have a linear magnitude response from dc to a frequency smaller than π

Simple FIR Digital Differentiators

First-Difference Differentiator is a first-order FIR discrete-time system with a time-domain input-output relation given by

$$y[n] = x[n] - x[n-1]$$

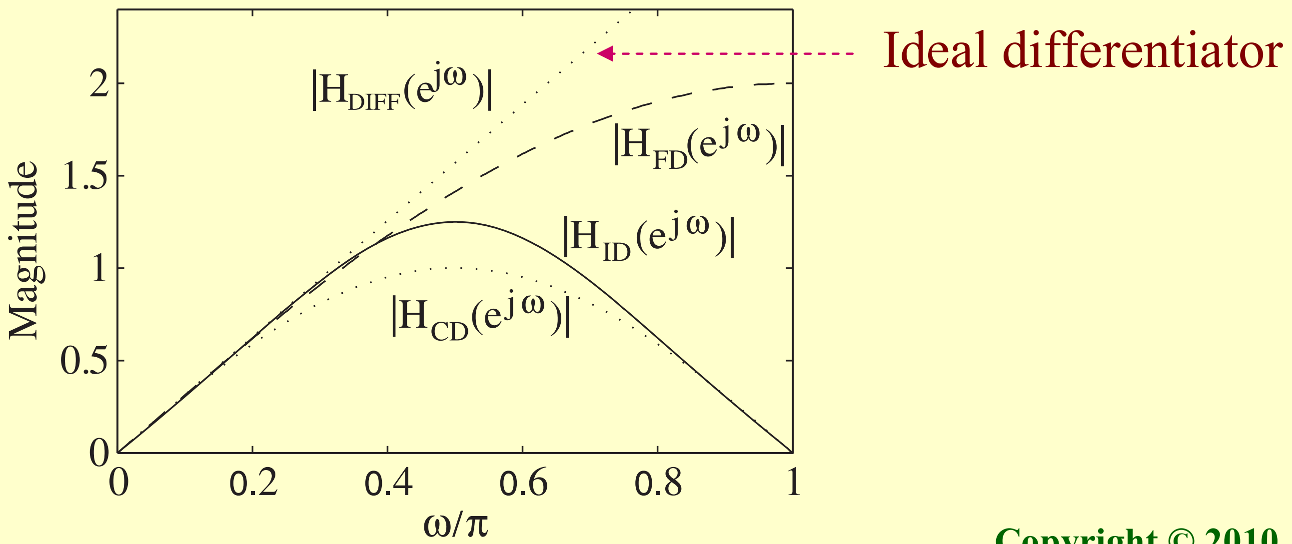
- Its transfer function is given by

$$H_{FD}(z) = 1 - z^{-1}$$

which is same as that of a first-order FIR highpass filter described earlier

Simple FIR Digital Differentiators

- Main drawback of the first-difference differentiator is that it also amplifies the high frequency noise often present in many signals



Simple FIR Digital Differentiators

Central-Difference Differentiator avoids the noise amplification problem of the first-difference differentiator

- Its time-domain input-output relation is

$$y[n] = \frac{1}{2}(x[n] - x[n-2])$$

- Its transfer function is given by

$$H_{CD}(z) = \frac{1}{2}(1 - z^{-2})$$

- It has a linear magnitude response in a very small low-frequency range

Can be derived also as the cascade of a basic lowpass and a basic highpass: h(n) = conv([1 1], [1 -1])

Higher-Order FIR Digital Differentiator

- The time-domain input-output relation of a higher-order FIR digital differentiator is given by

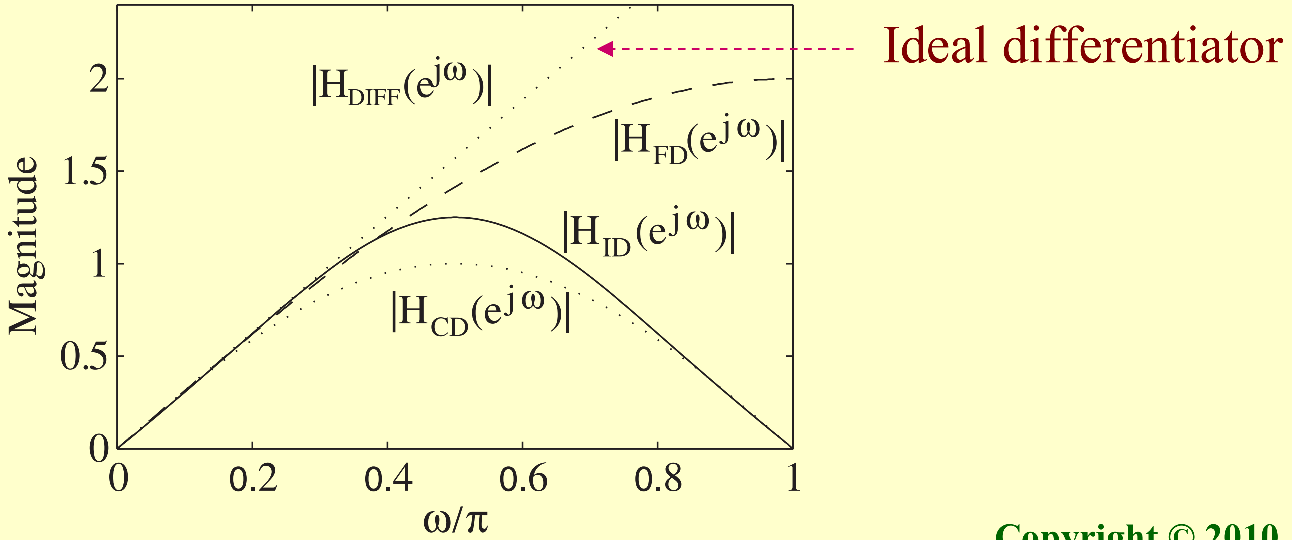
$$y[n] = -\tfrac{1}{16}x[n] + x[n-2] - x[n-4] + \tfrac{1}{16}x[n-6]$$

- Its transfer function is given by

$$H_{ID}(z) = -\tfrac{1}{16} + z^{-2} - z^{-4} + \tfrac{1}{16}z^{-6}$$

Higher-Order FIR Digital Differentiator

- Its magnitude response, scaled by a factor 0.6 is shown below
- The frequency range of operation of this differentiator is from dc to $\omega = 0.34\pi$



Comb Filters

- The simple filters discussed so far are characterized either by a single passband and/or a single stopband
- There are applications where filters with multiple passbands and stopbands are required
- The **comb filter** is an example of such filters

Comb Filters

- In its most general form, a comb filter has a frequency response that is a periodic function of ω with a period $2\pi/L$, where L is a positive integer
- If $H(z)$ is a filter with a single passband and/or a single stopband, a comb filter can be easily generated from it by replacing each delay in its realization with L delays resulting in a structure with a transfer function given by $G(z) = H(z^L)$

For ω in [0, 2π] we move L times along the unit circle

Comb Filters

- If $|H(e^{j\omega})|$ exhibits a peak at ω_p, then $|G(e^{j\omega})|$ will exhibit L peaks at $\omega_p k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- Likewise, if $|H(e^{j\omega})|$ has a notch at ω_o, then $|G(e^{j\omega})|$ will have L notches at $\omega_o k/L$, $0 \le k \le L-1$ in the frequency range $0 \le \omega < 2\pi$
- A comb filter can be generated from either an FIR or an IIR prototype filter

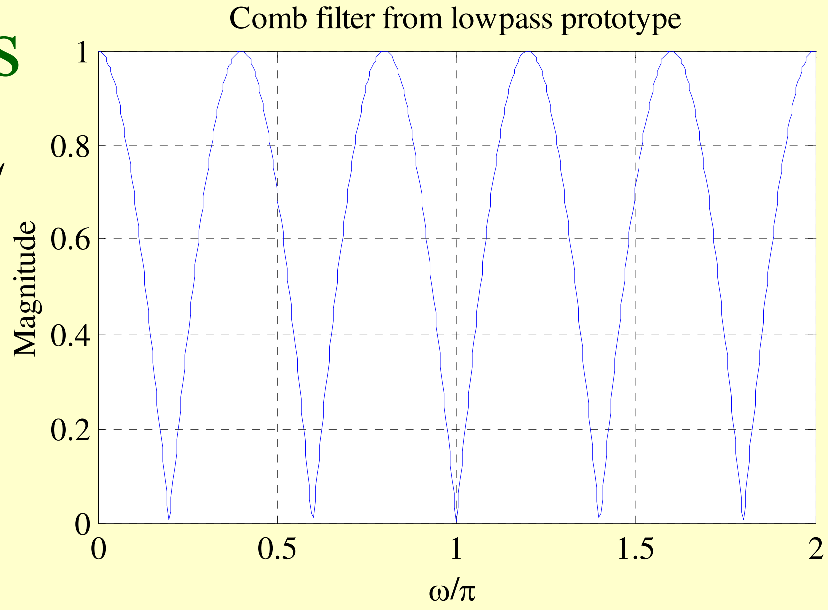
FIR Comb Filters

- For example, the comb filter generated from the prototype lowpass FIR filter $H_0(z) = \frac{1}{2}(1+z^{-1})$ has a transfer function

$$G_0(z) = H_0(z^L) = \frac{1}{2}(1+z^{-L})$$

Impulse response?

- $|G_0(e^{j\omega})|$ has L notches at $\omega = (2k+1)\pi/L$ and L peaks at $\omega = 2\pi\, k/L$, $0 \leq k \leq L-1$, in the frequency range $0 \leq \omega < 2\pi$

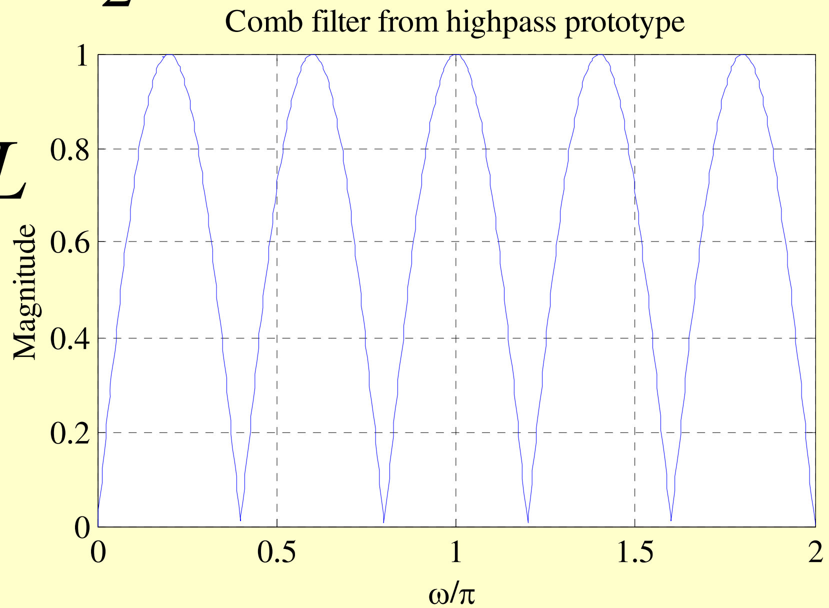


FIR Comb Filters

- For example, the comb filter generated from the prototype highpass FIR filter $H_1(z) = \frac{1}{2}(1 - z^{-1})$ has a transfer function

$$G_1(z) = H_1(z^L) = \tfrac{1}{2}(1 - z^{-L})$$

- $|G_1(e^{j\omega})|$ has L peaks at $\omega = (2k+1)\pi/L$ and L notches at $\omega = 2\pi\, k/L$, $0 \le k \le L-1$, in the frequency range $0 \le \omega < 2\pi$



FIR Comb Filters

- Depending on applications, comb filters with other types of periodic magnitude responses can be easily generated by appropriately choosing the prototype filter
- For example, the M-point moving average filter

$$H(z) = \frac{1 - z^{-M}}{M(1 - z^{-1})}$$

has been used as a prototype

FIR Comb Filters

- This filter has a peak magnitude at $\omega = 0$, and $M-1$ notches at $\omega = 2\pi\ell / M, 1 \le \ell \le M-1$
- The corresponding comb filter has a transfer function

$$G(z) = \frac{1-z^{-LM}}{M(1-z^{-L})}$$

whose magnitude has L peaks at $\omega = 2\pi k/L$, $0 \le k \le L-1$ and $L(M-1)$ notches at $\omega = 2\pi k/LM, 1 \le k \le L(M-1)$

IIR Comb Filters

- The transfer functions of the simplest forms of the prototype IIR filter are given by

$$H_0(z) = K\frac{1-z^{-1}}{1-\alpha z^{-1}}, \quad H_1(z) = K\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

  where $|\alpha| < 1$ for stability

- Note: $H_0(z)$ is a highpass filter with a zero at $z = 1$ and $H_1(z)$ is a lowpass filter with a zero at $z = -1$

IIR Comb Filters

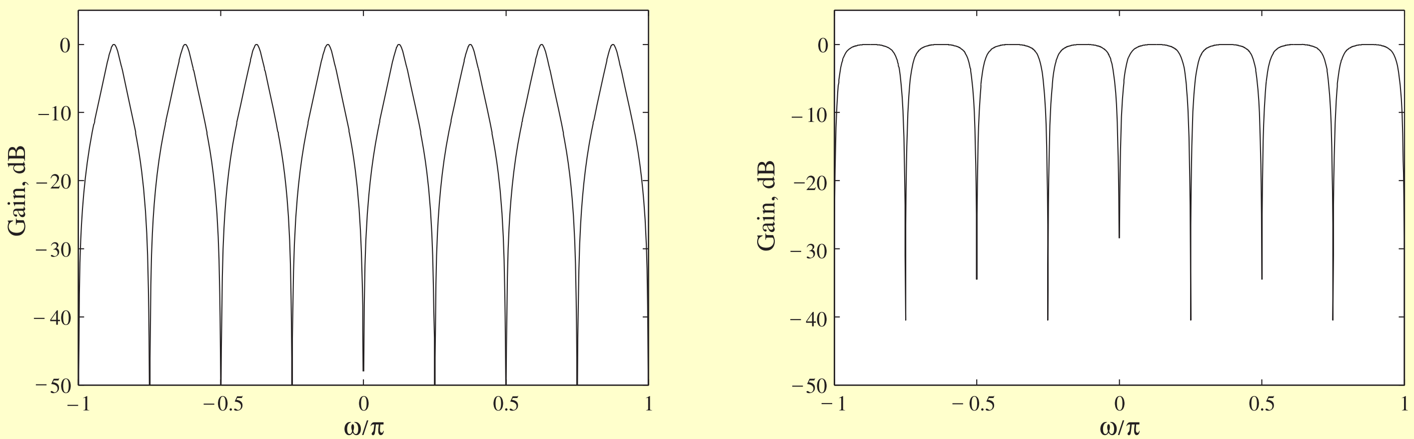
- For a maximum gain of 0 dB, the scale factor K of $H_0(z)$ should be set equal to $(1+\alpha)/2$ and the scale factor K of $H_1(z)$ should be set equal to $(1-\alpha)/2$
- The corresponding transfer functions of the comb filters of order L are

$$G_0(z) = K\frac{1-z^{-L}}{1-\alpha z^{-L}}, \quad G_1(z) = K\frac{1+z^{-L}}{1-\alpha z^{-L}}$$

Beware: poles get closer to the unit circle!

IIR Comb Filters

- Gain responses of the IIR comb filters generated from $H_0(z)$ and $H_1(z)$ for $L = 8$ are shown below



$G_0(z) = H_0(z^8)$ $\quad$ $G_1(z) = H_1(z^8)$

Complementary Transfer Functions

- A set of digital transfer functions with complementary characteristics often finds useful applications in practice
- useful complementary relations are described next along with some applications

Allpass Complementary Transfer Functions

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \le i \le M-1$, is defined to be **allpass-complementary** of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,

$$\sum_{i=0}^{M-1} H_i(z) = A(z)$$

Power-Complementary Transfer Functions

- A set of M digital transfer functions, $\{H_i(z)\}$, $0 \le i \le M-1$, is defined to be **power-complementary** of each other, if the sum of their square-magnitude responses is equal to a constant K for all values of ω, i.e.,

$$\sum_{i=0}^{M-1} \left| H_i(e^{j\omega}) \right|^2 = K, \qquad \text{for all } \omega$$

Power-Complementary Transfer Functions

- For a pair of power-complementary transfer functions, $H_0(z)$ and $H_1(z)$, the frequency ω_o where $|H_0(e^{j\omega_o})|^2 = |H_1(e^{j\omega_o})|^2 = 0.5$, is called the **cross-over frequency**
- At this frequency the gain responses of both filters are 3-dB below their maximum values
- As a result, ω_o is called the **3-dB cross-over frequency**

Power-Complementary Transfer Functions

- Example - Consider the two transfer functions $H_0(z)$ and $H_1(z)$ given by

$$H_0(z) = \tfrac{1}{2}[\mathcal{A}_0(z) + \mathcal{A}_1(z)]$$

$$H_1(z) = \tfrac{1}{2}[\mathcal{A}_0(z) - \mathcal{A}_1(z)]$$

where $\mathcal{A}_0(z)$ and $\mathcal{A}_1(z)$ are stable allpass transfer functions

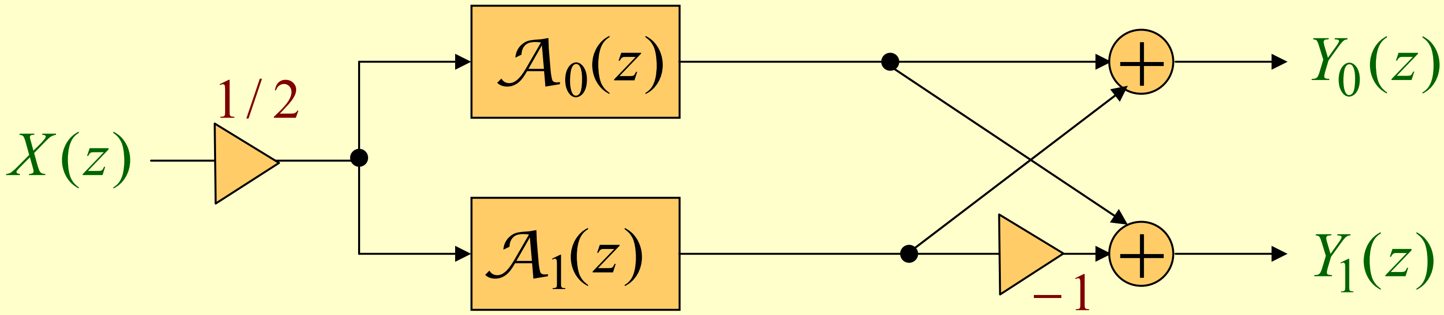
- Note that $H_0(z) + H_1(z) = \mathcal{A}_0(z)$
- Hence, $H_0(z)$ and $H_1(z)$ are allpass complementary

Power-Complementary Transfer Functions

- It can be shown that $H_0(z)$ and $H_1(z)$ are also power-complementary
- Moreover, $H_0(z)$ and $H_1(z)$ are bounded-real transfer functions

Doubly-Complementary Transfer Functions

- A pair of doubly-complementary IIR transfer functions, $H_0(z)$ and $H_1(z)$, with a sum of allpass decomposition can be simply realized as indicated below



$$H_0(z) = \frac{Y_0(z)}{X(z)} \qquad H_1(z) = \frac{Y_1(z)}{X(z)}$$

Doubly-Complementary Transfer Functions

- Example - The first-order lowpass transfer function

$$H_{LP}(z) = \frac{1-\alpha}{2}\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)$$

can be expressed as

$$H_{LP}(z) = \frac{1}{2}\left(1 + \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1}{2}[\mathcal{A}_0(z) + \mathcal{A}_1(z)]$$

where

$$\mathcal{A}_0(z) = 1, \quad \mathcal{A}_1(z) = \frac{-\alpha + z^{-1}}{1-\alpha z^{-1}}$$

Doubly-Complementary Transfer Functions

- Its power-complementary highpass transfer function is thus given by

$$H_{HP}(z) = \frac{1}{2}[\mathcal{A}_0(z) - \mathcal{A}_1(z)] = \frac{1}{2}\left(1 - \frac{-\alpha + z^{-1}}{1 - \alpha z^{-1}}\right)$$

$$= \frac{1+\alpha}{2}\left(\frac{1 - z^{-1}}{1 - \alpha z^{-1}}\right)$$

- The above expression is precisely the first-order highpass transfer function described earlier

Doubly-Complementary Transfer Functions

- Figure below demonstrates the allpass complementary property and the power complementary property of $H_{LP}(z)$ and $H_{HP}(z)$