Inverse systems
Recursive deconvolution
Sinusoidal oscillator

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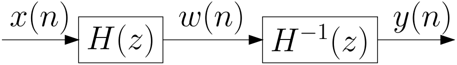
To recover a signal $x(n)$ that has undergone a distortion passing through a rational LTI system $H(z) = B(z)/A(z)$, one can try to determine the **inverse system** $H^{-1}(z) = A(z)/B(z)$:



– ideally, and apart from a delay, $y(n) = x(n)$.

It should be noted that the inverse system is completely determined only if its ROC is also defined. Usually we will be looking for a causal system, but in general different ROCs are obtained in different cases. E.g.:

$$H(z) = \frac{(z - 1/4)(z + 1/5)}{(z + 1/2)(z - 1/3)} \quad \longrightarrow \quad H^{-1}(z) = \frac{(z + 1/2)(z - 1/3)}{(z - 1/4)(z + 1/5)}$$

ROC of the inverse system:

– ROC_1 : $|z| > 1/4$ causal, stable;
– ROC_2 : $|z| < 1/5$ anticausal, unstable;
– ROC_3 : $1/5 < |z| < 1/4$ bilateral, unstable.

For a causal system, the inverse system can be built only if $H(z)$ is **minimum phase**: poles and zeros exchange their positions moving from H to H^{-1}, and if H is nonminimum phase H^{-1} is unstable.

Note that the outcome of the process can be severely affected

- by the presence of noise $d(n)$ added to $w(n)$,
- by an imprecise knowledge of $H(z)$,
- and by quantization errors in the computations.

A **recursive deconvolution** process can be devised too, that permits to estimate $x(n)$ without moving to the transform domain. The impulse response $h(n)$ of the FIR (or truncated IIR) system must be known in this case.

If $x(n)$ is causal, note that

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

For $n = 0$: $y(0) = x(0)h(0) \quad \longrightarrow \quad x(0) = \frac{y(0)}{h(0)}$;
for $n = 1$: $y(1) = x(0)h(1) + x(1)h(0) \quad \longrightarrow \quad x(1) = \frac{y(1) - x(0)h(1)}{h(0)}$;
for $n = 2$: $y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) \quad \longrightarrow$

$$x(2) = \frac{y(2) - x(1)h(1) - x(0)h(2)}{h(0)}.$$

In general, $x(n)$ can be derived rewriting the convolution sum as follows:

$$y(n) = x(n)h(0) + \sum_{k=0}^{n-1} x(k)h(n-k)$$

$$\longrightarrow \qquad x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)h(n-k)}{h(0)}$$

Note that errors may accumulate during the process.

MATLAB

Sinusoidal oscillator

We just need a pair of poles on the unit circle, and a trigger:

$$H(z) = \frac{1}{(1-pz^{-1})(1-p^*z^{-1})} = \frac{1}{1-(p+p^*)z^{-1}+pp^*z^{-2}}$$

$$= \frac{1}{1-2\cos\omega_0 z^{-1}+z^{-2}} \qquad (\text{since } p = e^{j\omega_0})$$

In the data domain:

$$y(n) = x(n) + 2\cos\omega_0 y(n-1) - y(n-2)$$

Trigger: Let $y(-1) = y(-2) = 0$, and $x(n) = A\sin\omega_0\delta(n)$. We get

$$y(n) = \{A\sin\omega_0,\ \ 2A\sin\omega_0\cos\omega_0 = A\sin 2\omega_0,\ \ A\sin 3\omega_0,\ \ldots\}$$

Suitable initial conditions permit to discard the input signal:

$$y(n) = 2\cos\omega_0 y(n-1) - y(n-2)$$

with $y(-1) = 0$ and $y(-2) = -A\sin\omega_0$.

see matlab and S04.4a
for effects of
quantization error