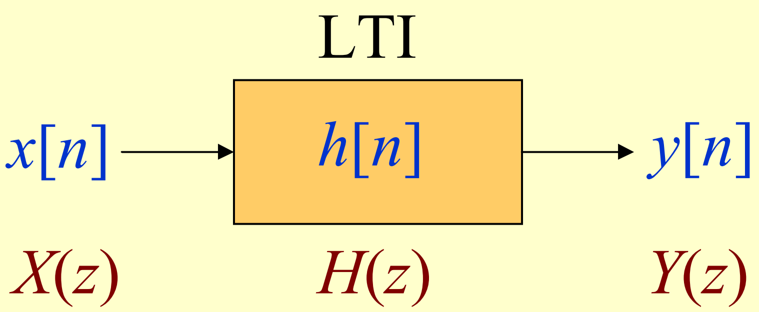
Types of Transfer Functions



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$Y(z) = H(z)X(z)$$

Types of Transfer Functions

- The time-domain classification of an LTI digital transfer function is based on the length of its impulse response $h[n]$:

  - Finite impulse response (FIR) transfer function

  - Infinite impulse response (IIR) transfer function

Types of Transfer Functions

- In the case of digital transfer functions with frequency-selective frequency responses, there are two types of classifications
- (1) Classification based on the shape of the magnitude function $|H(e^{j\omega})|$
- (2) Classification based on the the form of the phase function $\theta(\omega)$

Classification Based on Magnitude Characteristics

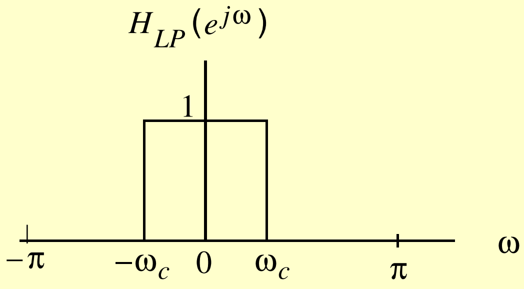
- One common classification is based on an ideal magnitude response
- A digital filter designed to pass signal components of certain frequencies without distortion should have a magnitude response equal to one at these frequencies, and should have a magnitude response equal to zero at all other frequencies

Ideal Filters

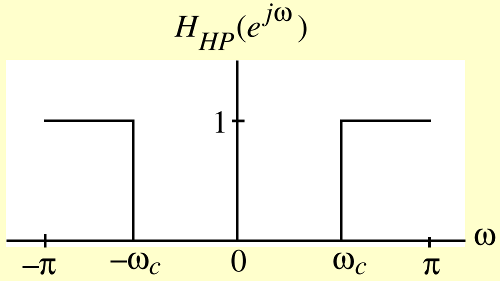
- The range of frequencies where the magnitude response takes the value of one is called the **passband**
- The range of frequencies where the magnitude response takes the value of zero is called the **stopband**

Ideal Filters

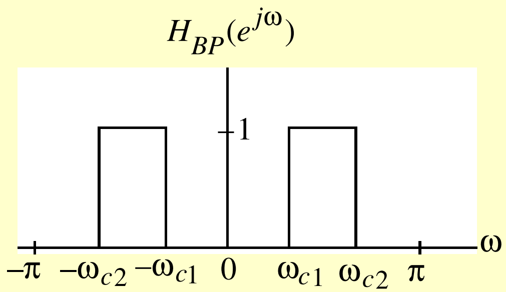
- Frequency responses of the four popular types of ideal digital filters with real impulse response coefficients are shown below:



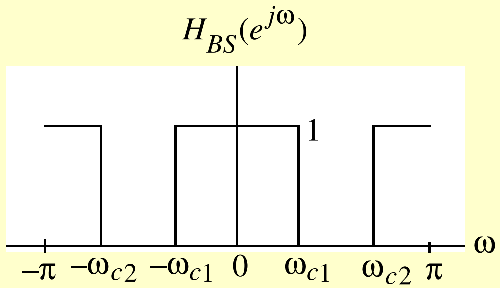
Lowpass



Highpass



Bandpass



Bandstop

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Ideal Filters

- Lowpass filter: Passband - $0 \le \omega \le \omega_c$

  Stopband - $\omega_c < \omega \le \pi$
- Highpass filter: Passband - $\omega_c \le \omega \le \pi$

  Stopband - $0 \le \omega < \omega_c$
- Bandpass filter: Passband - $\omega_{c1} \le \omega \le \omega_{c2}$

  Stopband - $0 \le \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi$
- Bandstop filter: Stopband - $\omega_{c1} < \omega < \omega_{c2}$

  Passband - $0 \le \omega \le \omega_{c1}$ and $\omega_{c2} \le \omega \le \pi$

Ideal Filters

- The frequencies ω_c , ω_{c1} , and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a zero phase everywhere

Ideal Filters

- Earlier in the course we derived the inverse DTFT of the frequency response $H_{LP}(e^{j\omega})$ of the ideal lowpass filter:

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- We have also shown that the above impulse response is not absolutely summable, and hence, the corresponding transfer function is not BIBO stable

Ideal Filters

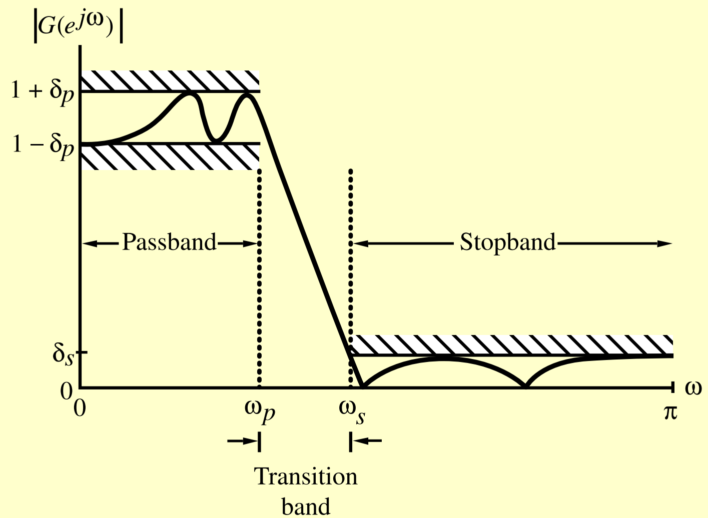
- Also, $h_{LP}[n]$ is not causal and is of doubly infinite length
- The remaining three ideal filters are also characterized by doubly infinite, noncausal impulse responses and are not absolutely summable
- Thus, the ideal filters with the ideal "brick wall" frequency responses cannot be realized with finite dimensional LTI filter

~~Ideal~~ Filters

- To develop stable and realizable transfer functions, the ideal frequency response specifications are relaxed by including a **transition band** between the passband and the stopband
- This permits the magnitude response to decay slowly from its maximum value in the passband to the zero value in the stopband

~~Ideal~~ Filters

- Moreover, the magnitude response is allowed to vary by a small amount both in the passband and the stopband
- Typical magnitude response specifications of a lowpass filter are shown below



Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ is defined as a **bounded real** (BR) **transfer function** if

$$|H(e^{j\omega})| \leq 1 \quad \text{for all values of } \omega$$

- Let $x[n]$ and $y[n]$ denote, respectively, the input and output of a digital filter characterized by a BR transfer function $H(z)$ with $X(e^{j\omega})$ and $Y(e^{j\omega})$ denoting their DTFTs

Bounded Real Transfer Functions

- Then the condition $|H(e^{j\omega})| \le 1$ implies that

$$\left|Y(e^{j\omega})\right|^2 \le \left|X(e^{j\omega})\right|^2$$

- Integrating the above from $-\pi$ to π, and applying Parseval's theorem we get

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \le \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

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Bounded Real Transfer Functions

- Thus, for all finite-energy inputs, the output energy is less than or equal to the input energy implying that a digital filter characterized by a BR transfer function can be viewed as a **passive structure**
- If $|H(e^{j\omega})| = 1$, then the output energy is equal to the input energy, and such a digital filter is therefore a **lossless system**

Bounded Real Transfer Functions

- A causal stable real-coefficient transfer function $H(z)$ with $|H(e^{j\omega})| = 1$ is thus called a **lossless bounded real (LBR) transfer function**
- The BR and LBR transfer functions are the keys to the realization of digital filters with low coefficient sensitivity

Bounded Real Transfer Functions

- Example – Consider the causal stable IIR transfer function

$$H(z) = \frac{K}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

where K and α are real constants

- Its square-magnitude function is given by

$$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

Bounded Real Transfer Functions

- The maximum value of $|H(e^{j\omega})|^2$ is obtained when $2\alpha\cos\omega$ in the denominator is a maximum and the minimum value is obtained when $2\alpha\cos\omega$ is a minimum
- For $\alpha > 0$, maximum value of $2\alpha\cos\omega$ is equal to 2α at $\omega = 0$, and minimum value is -2α at $\omega = \pi$

Bounded Real Transfer Functions

- Thus, for $\alpha > 0$, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = 0$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = \pi$
- On the other hand, for $\alpha < 0$, the maximum value of $2\alpha\cos\omega$ is equal to -2α at $\omega = \pi$ and the minimum value is equal to 2α at $\omega = 0$

Bounded Real Transfer Functions

- Here, the maximum value of $|H(e^{j\omega})|^2$ is equal to $K^2/(1-\alpha)^2$ at $\omega = \pi$ and the minimum value is equal to $K^2/(1+\alpha)^2$ at $\omega = 0$
- Hence, the maximum value can be made equal to 1 by choosing $K = \pm(1-\alpha)$, in which case the minimum value becomes $(1-\alpha)^2/(1+\alpha)^2$

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Bounded Real Transfer Functions

- Hence,

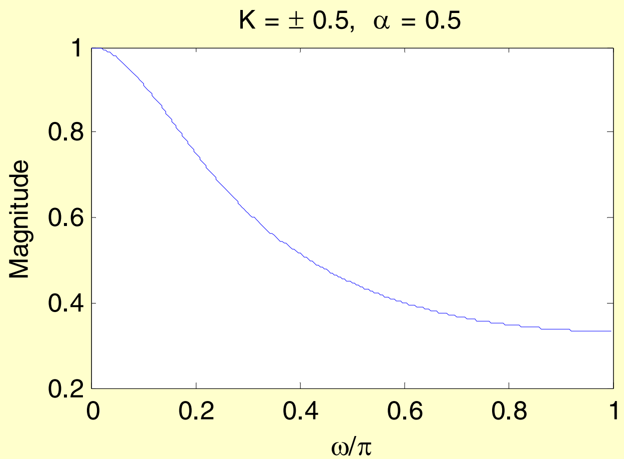
$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

is a BR function for $K = \pm(1-\alpha)$

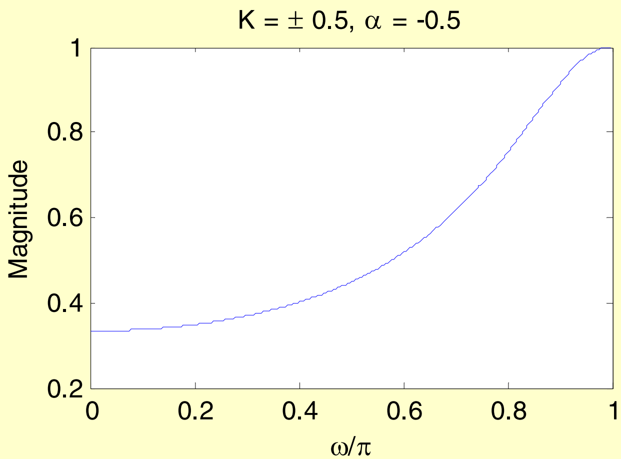
- Plots of the magnitude function for $\alpha = \pm 0.5$ with values of K chosen to make $H(z)$ a BR function are shown on the next slide

i.e.: y[n] is a weighted average (K>0)
or a "weighted difference" (K<0)
of x[n] and y[n-1]

Bounded Real Transfer Functions



Lowpass Filter



Highpass Filter

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Allpass Transfer Function

Definition

- An IIR transfer function $\mathcal{A}(z)$ with unity magnitude response for all frequencies, i.e.,

$$|\mathcal{A}(e^{j\omega})|^2 = 1, \qquad \text{for all } \omega$$

  is called an **allpass transfer function**

- An M-th order causal real-coefficient allpass transfer function is of the form

$$\mathcal{A}_M(z) = \pm \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

Allpass Transfer Function

- If we denote the denominator polynomial of $\mathcal{A}_M(z)$ as $D_M(z)$:

$$D_M(z) = 1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}$$

then it follows that $\mathcal{A}_M(z)$ can be written as:

$$\mathcal{A}_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Note from the above that if $z = re^{j\phi}$ is a pole of a real coefficient allpass transfer function, then it has a zero at $z = \frac{1}{r} e^{-j\phi}$

Allpass Transfer Function

- The numerator of a real-coefficient allpass transfer function is said to be the **mirror-image polynomial** of the denominator, and vice versa
- We shall use the notation $\tilde{D}_M(z)$ to denote the mirror-image polynomial of a degree-M polynomial $D_M(z)$, i.e.,

$$\tilde{D}_M(z) = z^{-M} D_M(z^{-1})$$

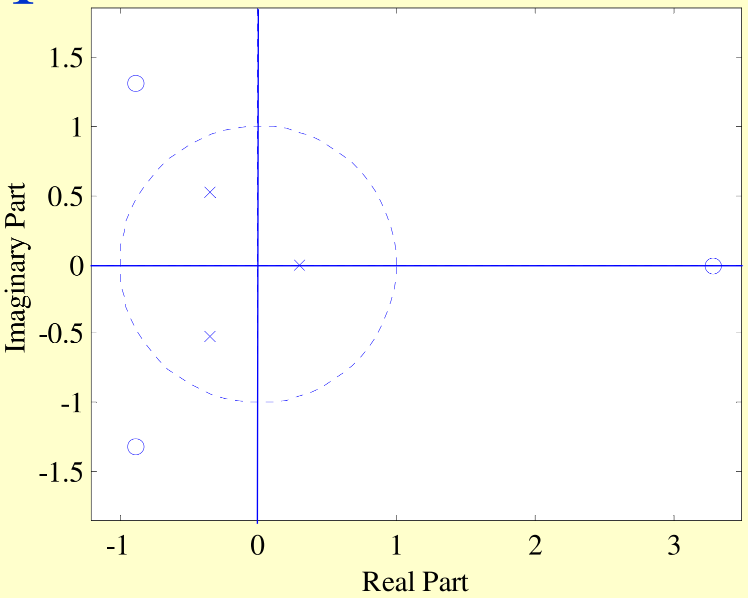
Allpass Transfer Function

- The expression

$$\mathcal{A}_M(z) = \pm \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

implies that the poles and zeros of a real-coefficient allpass function exhibit **mirror-image symmetry** in the *z*-plane

$$\mathcal{A}_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$



Allpass Transfer Function

- To show that $|\mathcal{A}_M(e^{j\omega})| = 1$ we observe that
$$\mathcal{A}_M(z^{-1}) = \pm \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$
- Therefore
$$\mathcal{A}_M(z)\mathcal{A}_M(z^{-1}) = \frac{z^{-M} D_M(z^{-1})}{D_M(z)} \frac{z^{M} D_M(z)}{D_M(z^{-1})}$$
- Hence
$$|\mathcal{A}_M(e^{j\omega})|^2 = \mathcal{A}_M(z)\mathcal{A}_M(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

Allpass Transfer Function

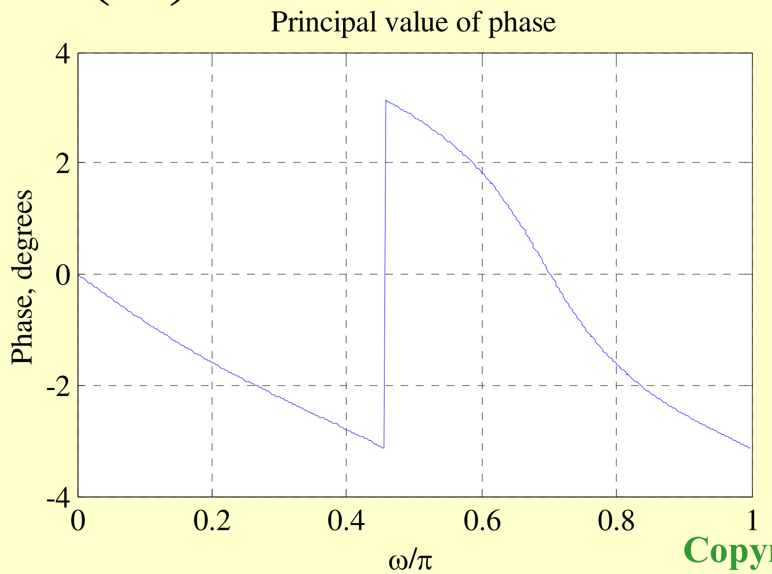
- Now, the poles of a causal stable transfer function must lie inside the unit circle in the z-plane
- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles situated inside the unit circle

Allpass Transfer Function

- Figure below shows the principal value of the phase of the 3rd-order allpass function

$$\mathcal{A}_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 0.4z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

- Note the discontinuity by the amount of 2π in the phase $\theta(\omega)$



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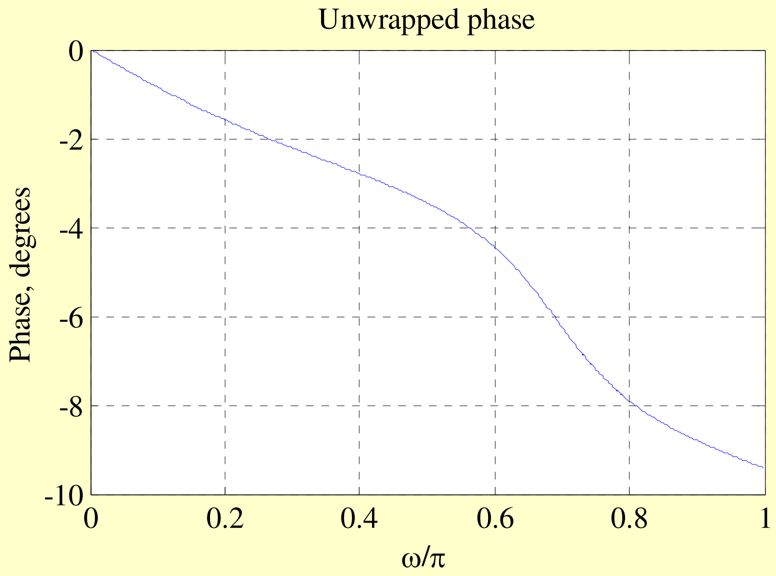
Allpass Transfer Function

- If we unwrap the phase by removing the discontinuity, we arrive at the unwrapped phase function $\theta_c(\omega)$ indicated below
- Note: The unwrapped phase function is a continuous function of ω



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Allpass Transfer Function

- The unwrapped phase function of any arbitrary causal stable allpass function is a continuous function of ω

Properties

- (1) A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) function or, equivalently, a causal stable allpass filter is a lossless structure

Allpass Transfer Function

- (2) The magnitude function of a stable allpass function $\mathcal{A}(z)$ satisfies:

$$|\mathcal{A}(z)| \begin{cases} <1, & \text{for } |z|>1 \\ =1, & \text{for } |z|=1 \\ >1, & \text{for } |z|<1 \end{cases}$$

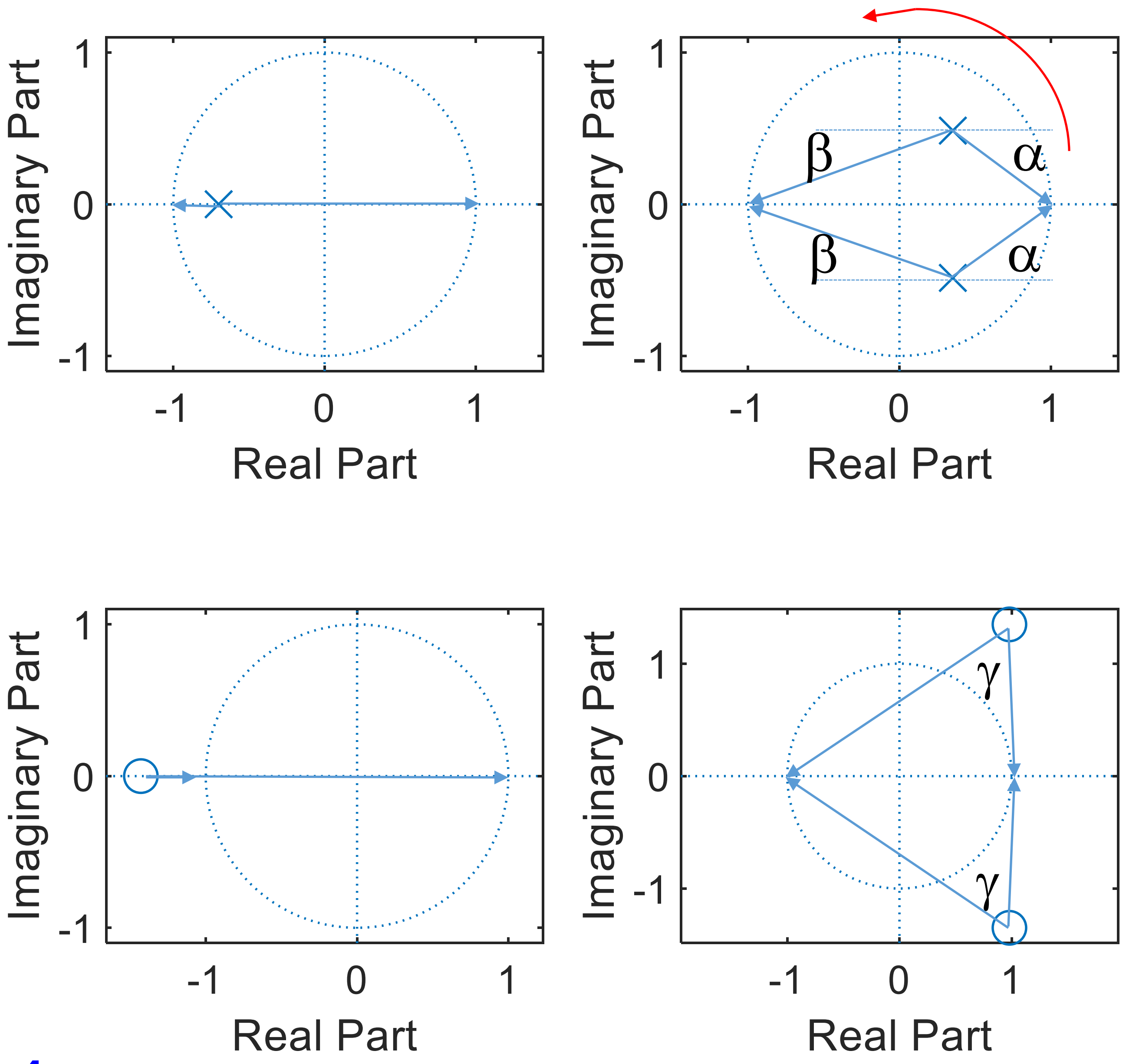
- (3) Let $\tau(\omega)$ denote the group delay function of an allpass filter $\mathcal{A}(z)$, i.e.,

$$\tau(\omega) = -\frac{d}{d\omega}[\theta_c(\omega)]$$

Allpass Transfer Function

- The unwrapped phase function $\theta_c(\omega)$ of a stable allpass function is a monotonically decreasing function of ω so that $\tau(\omega)$ is everywhere positive in the range $0 < \omega < \pi$
- The group delay of an M-th order stable real-coefficient allpass transfer function satisfies:

$$\int_0^{\pi} \tau(\omega) d\omega = M\pi$$



Phase rotation for $\omega = 0 \rightarrow \pi$:

Single pole:

$$\pi$$

Pole pair:

$$\pi+\alpha+\beta + \pi-\alpha-\beta = 2\pi$$

Single zero:

$$0$$

Zero pair:

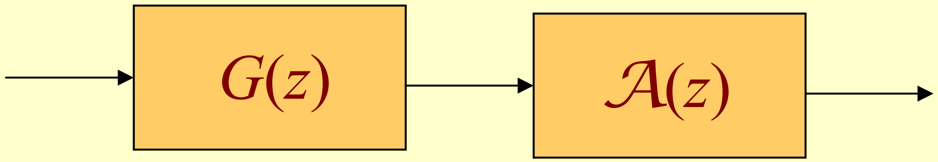
$$\gamma - \gamma = 0$$

Allpass Transfer Function

A Simple Application

- A simple but often used application of an allpass filter is as a **delay equalizer**
- Let $G(z)$ be the transfer function of a digital filter designed to meet a prescribed magnitude response
- The nonlinear phase response of $G(z)$ can be corrected by cascading it with an allpass filter $A(z)$ so that the overall cascade has a constant group delay in the band of interest

Allpass Transfer Function



- Since $|\mathcal{A}(e^{j\omega})| = 1$, we have

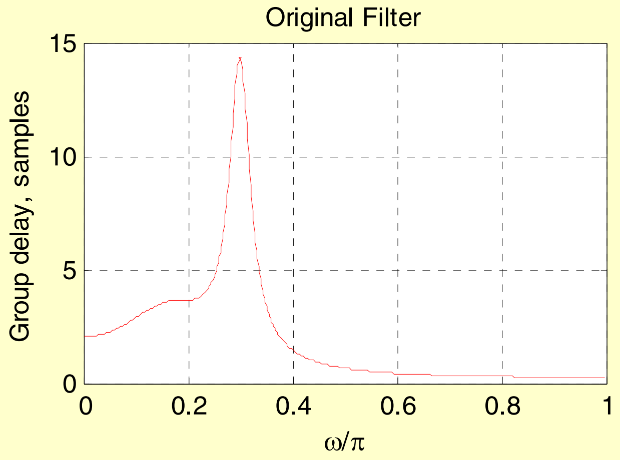
$$|G(e^{j\omega})\mathcal{A}(e^{j\omega})| = |G(e^{j\omega})|$$

- Overall group delay is the given by the sum of the group delays of $G(z)$ and $\mathcal{A}(z)$

Allpass Transfer Function

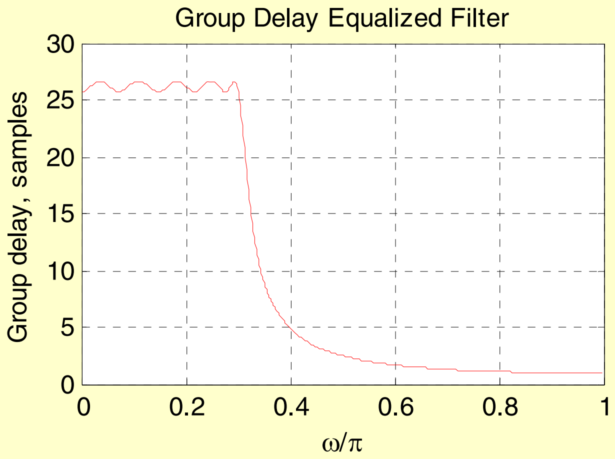
- Example – Figure below shows the group delay of a 4th order elliptic filter with the following specifications: $\omega_p = 0.3\pi$, $\delta_p = 1$ dB, $\delta_s = 35$ dB

(see slide 12)



Allpass Transfer Function

- Figure below shows the group delay of the original elliptic filter cascaded with an 8th order allpass section designed to equalize the group delay in the passband



MATLAB

Classification Based on Phase Characteristics

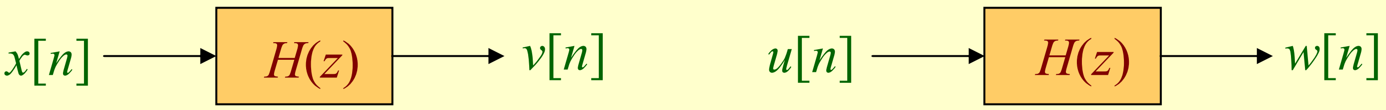
- A second classification of a transfer function is with respect to its phase characteristics
- In many applications, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in the passband

Zero-Phase Transfer Function

- One way to avoid any phase distortion is to make the frequency response of the filter real and nonnegative, i.e., to design the filter with a **zero phase characteristic**
- However, it is not possible to design a causal digital filter with a zero phase

Zero-Phase Transfer Function

- For non-real-time processing of real-valued input signals of finite length, zero-phase filtering can be very simply implemented by relaxing the causality requirement
- One zero-phase filtering scheme is sketched below

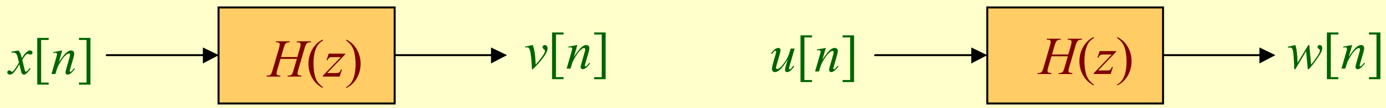


$$u[n] = v[-n], \qquad y[n] = w[-n]$$

Zero-Phase Transfer Function

- It is easy to verify the above scheme in the frequency domain
- Let $X(e^{j\omega})$, $V(e^{j\omega})$, $U(e^{j\omega})$, $W(e^{j\omega})$, and $Y(e^{j\omega})$ denote the DTFTs of $x[n]$, $v[n]$, $u[n]$, $w[n]$, and $y[n]$, respectively
- From the figure shown earlier and making use of the symmetry relations we arrive at the relations between various DTFTs as given on the next slide

Zero-Phase Transfer Function



$$u[n] = v[-n], \qquad y[n] = w[-n]$$

$$V(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \qquad W(e^{j\omega}) = H(e^{j\omega})U(e^{j\omega})$$

$$U(e^{j\omega}) = V^*(e^{j\omega}), \qquad Y(e^{j\omega}) = W^*(e^{j\omega})$$

- Combining the above equations we get

$$Y(e^{j\omega}) = W^*(e^{j\omega}) = H^*(e^{j\omega})U^*(e^{j\omega})$$

$$= H^*(e^{j\omega})V(e^{j\omega}) = H^*(e^{j\omega})\,H(e^{j\omega})\,X(e^{j\omega})$$

$$= \left|H(e^{j\omega})\right|^2 X(e^{j\omega})$$

Note: Replacing a desired H with |H|^2 is ok for many specifications like the one in slide 12

Zero-Phase Transfer Function

- The function `filtfilt` implements the above zero-phase filtering scheme
- In the case of a causal transfer function with a nonzero phase response, the phase distortion can be avoided by ensuring that the transfer function has a unity magnitude and a **linear-phase** characteristic in the frequency band of interest

Linear-Phase Transfer Function

- A full-band filter with a linear phase has a frequency response given by

$$H(e^{j\omega}) = e^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$

- Note also $\left|H(e^{j\omega})\right| = 1$

$$\tau(\omega) = D$$

Linear-Phase Transfer Function

- The output $y[n]$ of this filter to an input $x[n] = Ae^{j\omega n}$ is then given by

$$y[n] = Ae^{-j\omega D}e^{j\omega n} = Ae^{j\omega(n-D)}$$

- If $x_a(t)$ and $y_a(t)$ represent the continuous-time signals whose sampled versions, sampled at $t = nT$, are $x[n]$ and $y[n]$ given above, then the delay between $x_a(t)$ and $y_a(t)$ is precisely the group delay of amount D

Linear-Phase Transfer Function

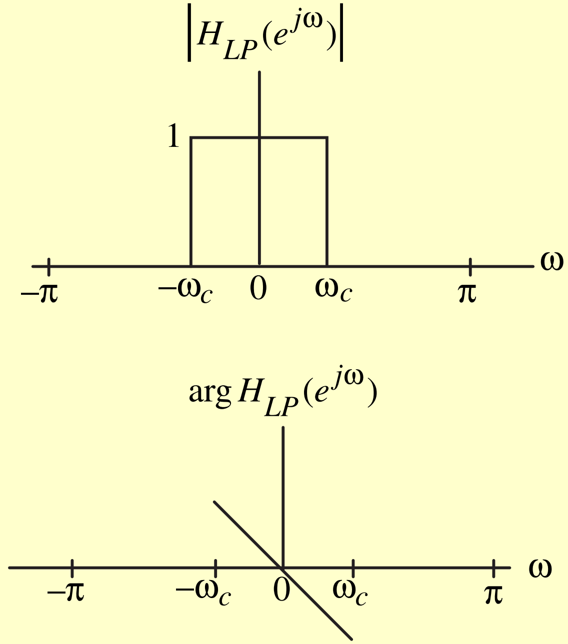
- If D is an integer, then $y[n]$ is identical to $x[n]$, but delayed by D samples
- If D is not an integer, $y[n]$, being delayed by a fractional part, is not identical to $x[n]$
- In the latter case, the waveform of the underlying continuous-time output is identical to the waveform of the underlying continuous-time input and delayed D units of time

Linear-Phase Transfer Function

- If it is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase, then the transfer function should exhibit a unity magnitude response and a linear-phase response in the band of interest

Linear-Phase Transfer Function

- Figure below shows the frequency response of a lowpass filter with a linear-phase characteristic in the passband



Linear-Phase Transfer Function

- Since the signal components in the stopband are blocked, the phase response in the stopband can be of any shape
- Example - Determine the impulse response of an ideal lowpass filter with a linear phase response:

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega n_o}, & 0 < |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

Linear-Phase Transfer Function

- Applying the frequency-shifting property of the DTFT to the impulse response of an ideal zero-phase lowpass filter we arrive at

$$h_{LP}[n] = \frac{\sin \omega_c (n - n_o)}{\pi (n - n_o)}, \quad -\infty < n < \infty$$

- As before, the above filter is noncausal and of doubly infinite length, and hence, unrealizable

Linear-Phase Transfer Function

- By truncating the impulse response to a finite number of terms, a realizable FIR approximation to the ideal lowpass filter can be developed
-

Linear-Phase Transfer Function

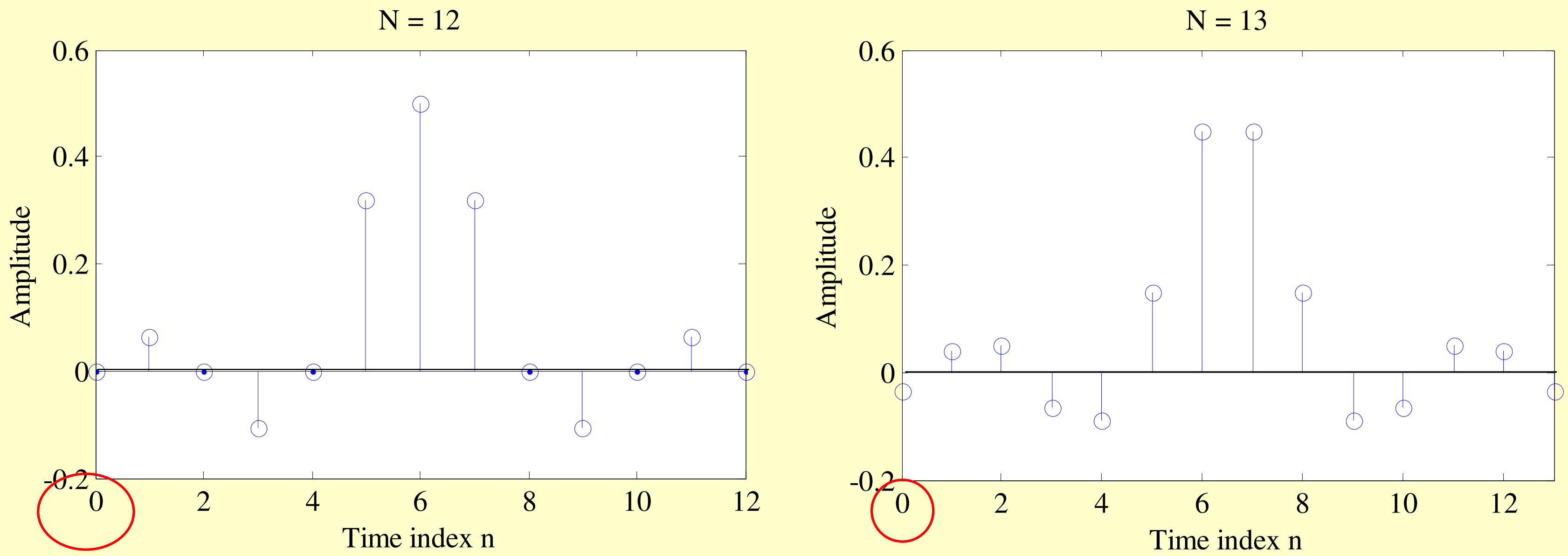
- If we choose $n_o = N/2$ with N a positive integer, the truncated and shifted approximation

$$\hat{h}_{LP}[n] = \frac{\sin \omega_c (n - N/2)}{\pi(n - N/2)}, \quad 0 \le n \le N$$

will be a length N+1 causal linear-phase FIR filter

Linear-Phase Transfer Function

- Figure below shows the filter coefficients obtained using the function `sinc` for two different values of N



~~Zero-Phase Response~~

- Because of the symmetry of the impulse response coefficients as indicated in the two figures, the frequency response of the truncated approximation can be expressed as:

$$\hat{H}_{LP}(e^{j\omega}) = \sum_{n=0}^{N} \hat{h}_{LP}[n] e^{-j\omega n} = e^{-j\omega N/2} \tilde{H}_{LP}(\omega)$$

where $\tilde{H}_{LP}(\omega)$, called the ~~**zero-phase response**~~ or **amplitude response**, is a real function of ω

(Further design details in Sec.04.5)

Minimum-Phase and Maximum-Phase Transfer Functions

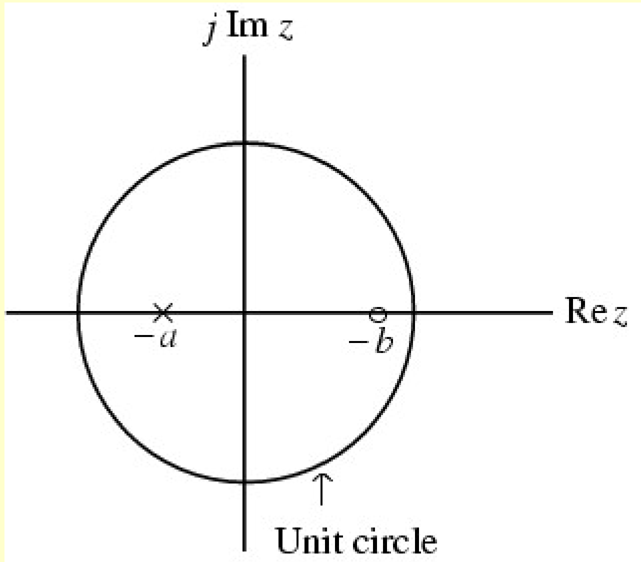
- Consider the two 1st-order transfer functions:

$$H_1(z) = \frac{z+b}{z+a}, \quad H_2(z) = \frac{bz+1}{z+a}, \quad |a| < 1, \quad |b| < 1$$

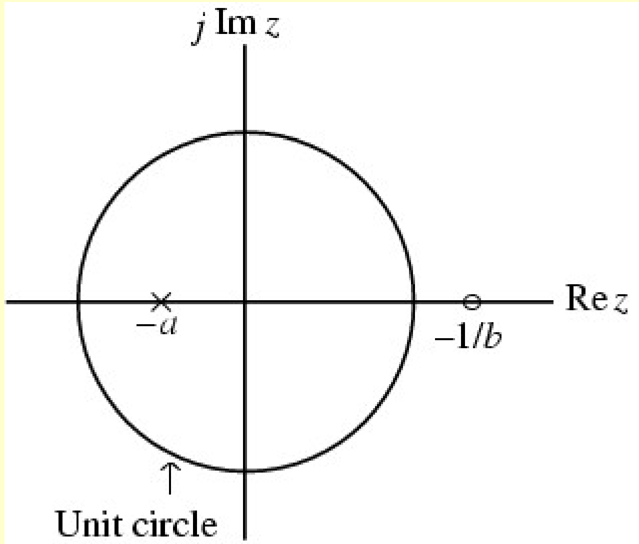
- Both transfer functions have a pole inside the unit circle at the same location $z = -a$ and are stable
- But the zero of $H_1(z)$ is inside the unit circle at $z = -b$, whereas, the zero of $H_2(z)$ is at $z = -\frac{1}{b}$ situated in a mirror-image symmetry

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the pole-zero plots of the two transfer functions



$H_1(z)$



$H_2(z)$

Note: numerator in H_2 should more properly be $(z + 1/b)$. A gain factor is introduced in the notation used above

Minimum-Phase and Maximum-Phase Transfer Functions

- However, both transfer functions have an identical magnitude function as

$$H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1})$$

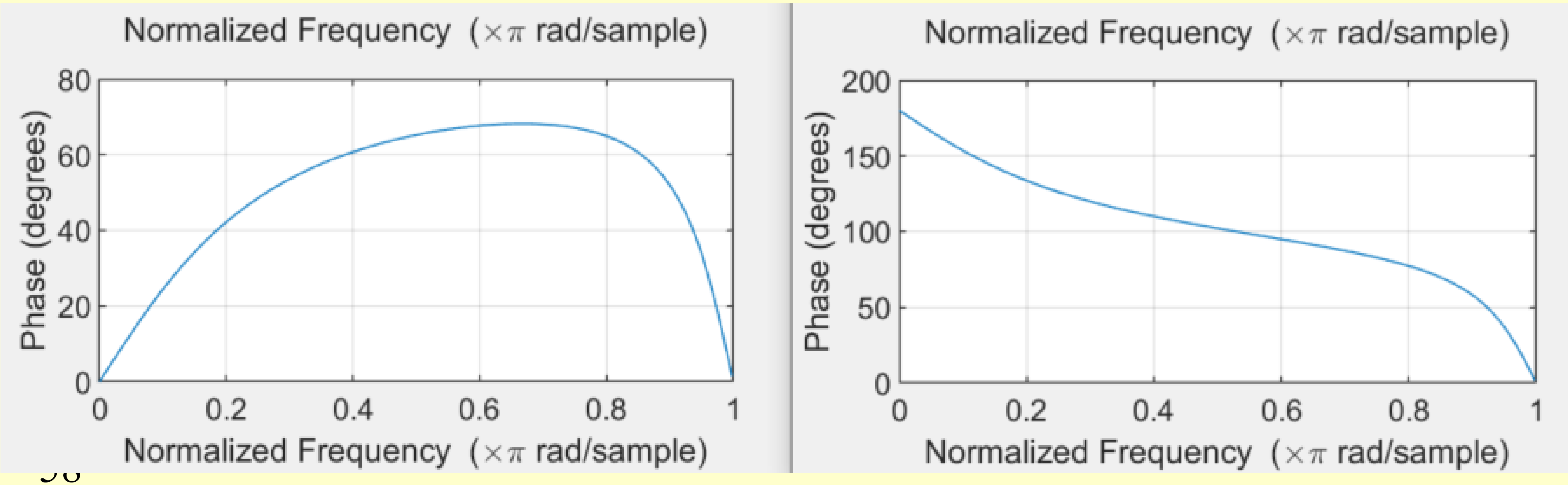
- The corresponding phase functions are

$$\arg[H_1(e^{j\omega})] = \tan^{-1}\frac{\sin\omega}{b+\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

$$\arg[H_2(e^{j\omega})] = \tan^{-1}\frac{b\sin\omega}{1+b\cos\omega} - \tan^{-1}\frac{\sin\omega}{a+\cos\omega}$$

Minimum-Phase and Maximum-Phase Transfer Functions

- Figure below shows the unwrapped phase responses of the two transfer functions for $a = 0.8$ and $b = -0.5$



Minimum-Phase and Maximum-Phase Transfer Functions

- From this figure it follows that $H_2(z)$ has an excess phase lag with respect to $H_1(z)$
- The excess phase lag property of $H_2(z)$ with respect to $H_1(z)$ can also be explained by observing that we can write

$$H_2(z) = \frac{bz+1}{z+a} = \underbrace{\left(\frac{z+b}{z+a}\right)}_{H_1(z)}\underbrace{\left(\frac{bz+1}{z+b}\right)}_{A(z)}$$

Minimum-Phase and Maximum-Phase Transfer Functions

where $\mathcal{A}(z) = (bz+1)/(z+b)$ is a stable allpass function

- The phase functions of $H_1(z)$ and $H_2(z)$ are thus related through

$$arg[H_2(e^{j\omega})] = arg[H_1(e^{j\omega})] + arg[\mathcal{A}(e^{j\omega})]$$

- As the unwrapped phase function of a stable first-order allpass function is a negative function of ω, it follows from the above that $H_2(z)$ has indeed an excess phase lag with respect to $H_1(z)$

Minimum-Phase and Maximum-Phase Transfer Functions

- Generalizing the above result, let $H_m(z)$ be a causal stable transfer function with all zeros inside the unit circle and let $H(z)$ be another causal stable transfer function satisfying $\left|H(e^{j\omega})\right| = \left|H_m(e^{j\omega})\right|$
- These two transfer functions are then related through $H(z) = H_m(z)\mathcal{A}(z)$ where $\mathcal{A}(z)$ is a causal stable allpass function

Minimum-Phase and Maximum-Phase Transfer Functions

- The unwrapped phase functions of $H_m(z)$ and $H(z)$ are thus related through

$$arg[H(e^{j\omega})] = arg[H_m(e^{j\omega})] + arg[\mathcal{A}(e^{j\omega})]$$

- $H(z)$ has an excess phase lag with respect to $H_m(z)$
- A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function**

Minimum-Phase and Maximum-Phase Transfer Functions

- A causal stable transfer function with all zeros outside the unit circle is called a **maximum-phase transfer function**
- A causal stable transfer function with zeros inside and outside the unit circle is called a **mixed-phase transfer function**

Minimum-Phase and Maximum-Phase Transfer Functions

- Example – Consider the mixed-phase transfer function

$$H(z)=\frac{2(1+0.3z^{-1})(0.4-z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}$$

- We can rewrite $H(z)$ as

$$H(z)=\underbrace{\left[\frac{2(1+0.3z^{-1})(1-0.4z^{-1})}{(1-0.2z^{-1})(1+0.5z^{-1})}\right]}_{\text{Minimum-phase function}}\underbrace{\left(\frac{0.4-z^{-1}}{1-0.4z^{-1}}\right)}_{\text{Allpass function}}$$