LTI Discrete-Time Systems in the z-Transform Domain

- An LTI discrete-time system is completely characterized in the time-domain by its impulse response sequence $\{h[n]\}$
- Thus, the transform-domain representation of a discrete-time signal can also be equally applied to the transform-domain representation of an LTI discrete-time system

LTI Discrete-Time Systems in the $\boxed{z\text{-Transform}}$ Domain

- Such transform-domain representations provide additional insight into the behavior of such systems
- It is easier to design and implement these systems in the transform-domain for certain applications
- We consider now the use of the DTFT and the *z*-transform in developing the transform-domain representations of an LTI system

Finite-Dimensional LTI Discrete-Time Systems

- In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

Finite-Dimensional LTI Discrete-Time Systems

- Applying the z-transform to both sides of the difference equation and making use of the linearity and the time-invariance properties of Table ~~6.2~~ we arrive at

$$\sum_{k=0}^{N} d_k z^{-k} Y(z) = \sum_{k=0}^{M} p_k z^{-k} X(z)$$

where $Y(z)$ and $X(z)$ denote the z-transforms of $y[n]$ and $x[n]$ with associated ROCs, respectively

Finite-Dimensional LTI Discrete-Time Systems

- A more convenient form of the z-domain representation of the difference equation is given by

$$\left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)$$

The Transfer Function

- A generalization of the frequency response function
- The convolution sum description of an LTI discrete-time system with an impulse response $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

The Transfer Function

- Taking the z-transforms of both sides we get

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n] z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k] x[n-k] \right) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k] z^{-n} \right)$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+k)} \right)$$

The Transfer Function

- Or, $Y(z) = \sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} \right)}_{X(z)} z^{-k}$
- Therefore, $Y(z) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right)}_{H(z)} X(z)$
- Thus, $Y(z) = H(z)X(z)$

Practically usable only for **FIR** systems

The Transfer Function

- Hence,

$$H(z) = Y(z)/X(z)$$

- The function $H(z)$, which is the z-transform of the impulse response $h[n]$ of the LTI system, is called the **transfer function** or the **system function**
- The inverse z-transform of the transfer function $H(z)$ yields the impulse response $h[n]$

The Transfer Function

- Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Its transfer function is obtained by taking the z-transform of both sides of the above equation
- Thus

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

Transfer function for **IIR** systems

The Transfer Function

- Or, equivalently as

$$H(z) = z^{(N-M)} \frac{\sum_{k=0}^{M} p_k z^{M-k}}{\sum_{k=0}^{N} d_k z^{N-k}}$$

- An alternate form of the transfer function is given by

$$H(z) = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^{M} (1 - \xi_k z^{-1})}{\prod_{k=1}^{N} (1 - \lambda_k z^{-1})}$$

The Transfer Function

- Or, equivalently as

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

- $\xi_1, \xi_2, \ldots, \xi_M$ are the finite **zeros**, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the finite **poles** of $H(z)$
- If $N > M$, there are additional $(N-M)$ zeros at $z = 0$
- If $N < M$, there are additional $(M-N)$ poles at $z = 0$

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The Transfer Function

- For a causal IIR digital filter, the impulse response is a causal sequence
- The ROC of the causal transfer function

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

is thus exterior to a circle going through the pole furthest from the origin

- Thus the ROC is given by $|z| > \max_k |\lambda_k|$

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The Transfer Function

- Example - Consider the M-point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M[z^{M-1}(z-1)]}$$

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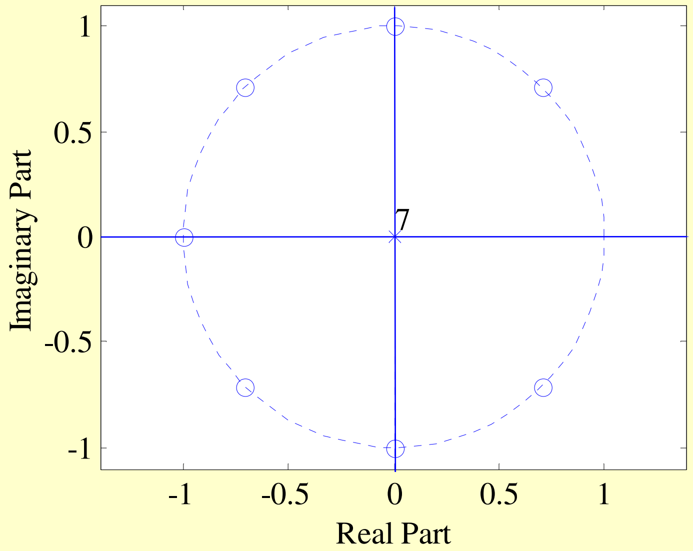
The Transfer Function

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \le k \le M-1$
- There are $M-1$ poles at $z = 0$ and a single pole at $z = 1$
- The pole at $z = 1$ exactly cancels the zero at z = 1
- The ROC is the entire z-plane except $z = 0$

$M = 8$



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The Transfer Function

- Example - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n] = x[n-1] - 1.2\,x[n-2] + x[n-3] + 1.3\,y[n-1] - 1.04\,y[n-2] + 0.222\,y[n-3]$$

- Its transfer function is therefore given by

$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$

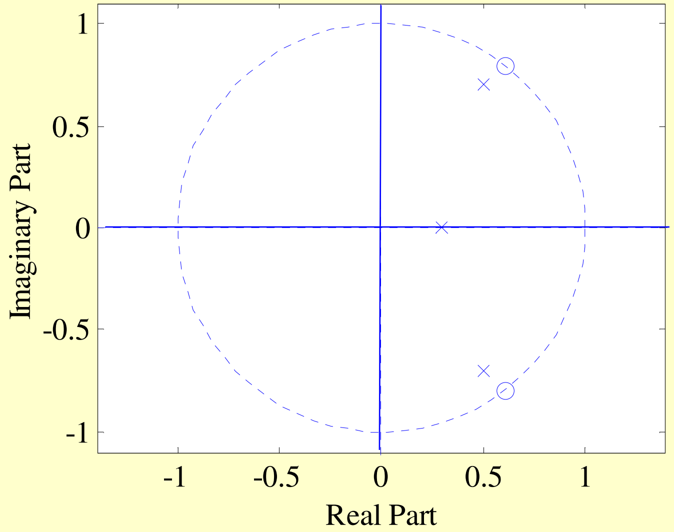
The Transfer Function

- Alternate forms:

$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$$

$$= \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$
- ROC: $|z| > \sqrt{0.74}$



Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that

$$\boxed{\begin{aligned} \left|H(e^{j\omega})\right|^2 &= H(e^{j\omega})H^*(e^{j\omega}) \\ &= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}} \end{aligned}}$$

Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)}$$

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Frequency Response from Transfer Function

- It is convenient to visualize the contributions of the **zero factor** $(z-\xi_k)$ and the **pole factor** $(z-\lambda_k)$ from the factored form of the frequency response
- The magnitude function is given by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

Frequency Response from Transfer Function

which reduces to

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M)$$
$$+\sum_{k=1}^{M}\arg(e^{j\omega}-\xi_k) - \sum_{k=1}^{N}\arg(e^{j\omega}-\lambda_k)$$

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Frequency Response from Transfer Function

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{p_0}{d_0}\right|^2 \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)(e^{-j\omega}-\xi_k^*)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)(e^{-j\omega}-\lambda_k^*)}$$

Matlab 04_2 and 04_3: examples in the Laplace and *z* domains

Geometric Interpretation of Frequency Response Computation

- The factored form of the frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

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Geometric Interpretation of Frequency Response Computation

- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency
- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

where $\rho e^{j\phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

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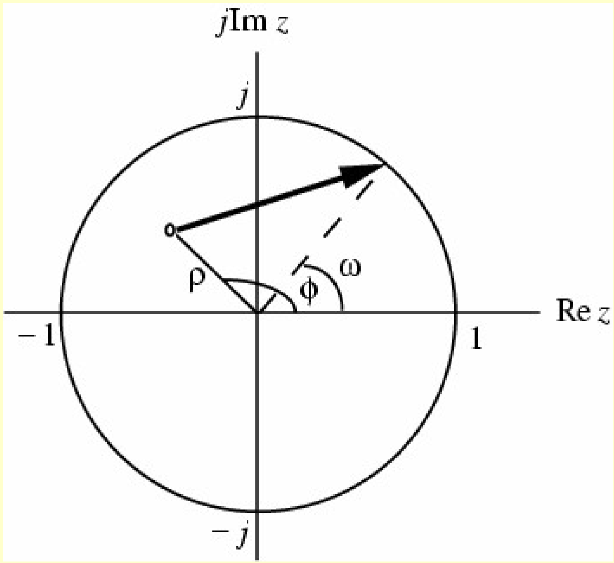
Geometric Interpretation of Frequency Response Computation

- As shown below in the z-plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



Geometric Interpretation of Frequency Response Computation

- As ω is varied from 0 to 2π, the tip of the vector moves counterclockise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

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Geometric Interpretation of Frequency Response Computation

- As indicated by

$$|H(e^{j\omega})| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M} |e^{j\omega} - \xi_k|}{\prod_{k=1}^{N} |e^{j\omega} - \lambda_k|}$$

the magnitude response $|H(e^{j\omega})|$ at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

Geometric Interpretation of Frequency Response Computation

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

  we observe that the phase response at a specific value of ω is obtained by adding the phase of the term p_0/d_0 and the linear-phase term $\omega(N-M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

Geometric Interpretation of Frequency Response Computation

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$

Geometric Interpretation of Frequency Response Computation

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range
- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to ~~or on~~ the unit circle in this range

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Stability Condition in Terms of the Pole Locations

- A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

Stability Condition in Terms of the Pole Locations

- The ROC of the z-transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa

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Stability Condition in Terms of the Pole Locations

- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z = \infty$
- An FIR digital filter with bounded impulse response is always stable

Stability Condition in Terms of the Pole Locations

- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

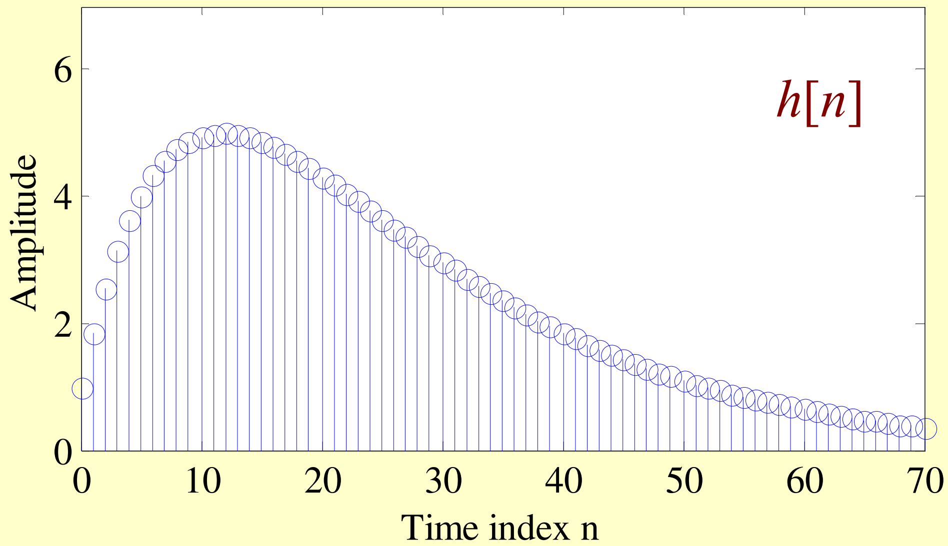
Stability Condition in Terms of the Pole Locations

- Example - Consider the causal IIR transfer function

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

Stability Condition in Terms of the Pole Locations



- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases

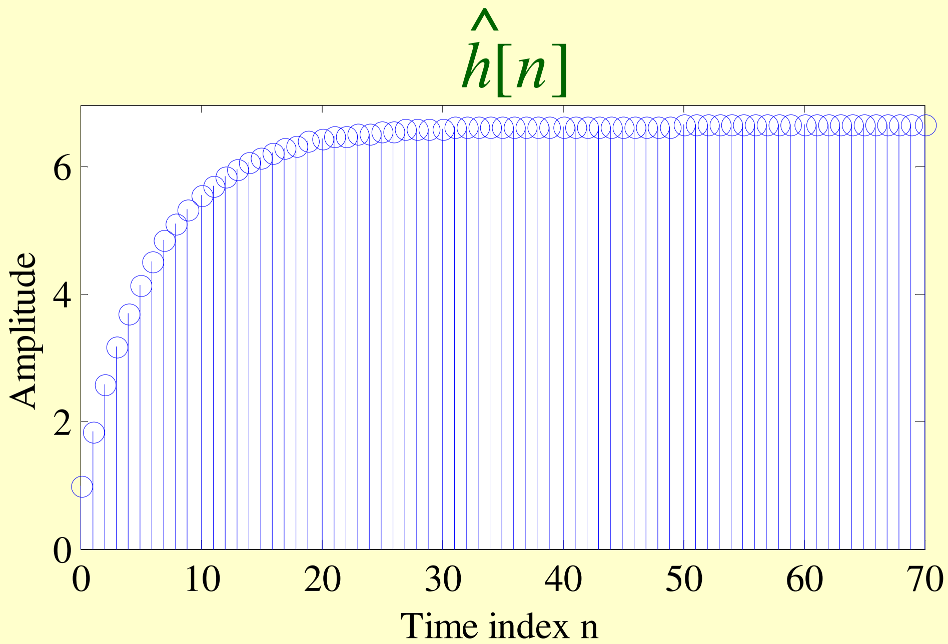
Stability Condition in Terms of the Pole Locations

- The absolute summability condition of $h[n]$ is satisfied
- Hence, $H(z)$ is a stable transfer function
- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

Stability Condition in Terms of the Pole Locations

- A plot of the impulse response of $\hat{h}[n]$ is shown below



Stability Condition in Terms of the Pole Locations

- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases
- Hence, the absolute summability condition of is violated
- Thus, $\hat{H}(z)$ is an unstable transfer function

Stability Condition in Terms of the Pole Locations

- The stability testing of a IIR transfer function is therefore an important problem
- In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- For a causal IIR transfer function, the sum S can be computed approximately as

$$S_K = \sum_{n=0}^{K-1} |h[n]|$$

Stability Condition in Terms of the Pole Locations

- The partial sum is computed for increasing values of K until the difference between a series of consecutive values of S_K is smaller than some arbitrarily chosen small number, which is typically 10^{-6}
- For a transfer function of very high order this approach may not be satisfactory
- An alternate, easy-to-test, stability condition is developed next

Stability Condition in Terms of the Pole Locations

- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- Its impulse response $\{h[n]\}$ is a right-sided sequence
- The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$

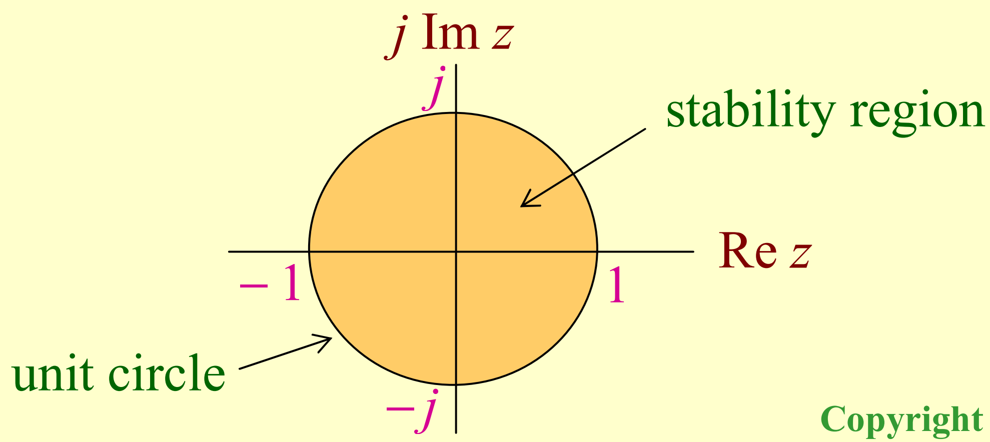
Stability Condition in Terms of the Pole Locations

- But stability requires that $\{h[n]\}$ be absolutely summable
- This in turn implies that the DTFT $H(e^{j\omega})$ of $\{h[n]\}$ exists
- Now, if the ROC of the z-transform $H(z)$ includes the unit circle, then

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Stability Condition in Terms of the Pole Locations

- Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle
- The stability region (shown shaded) in the z-plane is shown below



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Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$H(z) = \frac{1}{1-0.845z^{-1}+0.850586z^{-2}}$$

is

$$H(z) = \frac{1}{(1-0.902z^{-1})(1-0.943z^{-1})}$$

which has a real pole at $z = 0.902$ and a real pole at $z = 0.943$

- Since both poles are inside the unit circle, $H(z)$ is BIBO stable

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Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85 z^{-1} + 0.85 z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85 z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since both poles are not inside the unit circle, $H(z)$ is unstable

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