The Frequency Response

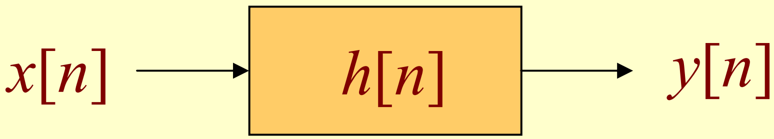
- Discrete-time signals encountered in practice can be represented as a linear combination of a very large, maybe infinite, number of sinusoidal discrete-time signals of different angular frequencies
- Thus, knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of the superposition property

Eigen Function

- An important property of an LTI system is that for certain types of input signals, called **eigen functions**, the output signal is the input signal multiplied by a complex constant
- We consider here one such eigen function as the input

Eigen Function

- Consider the LTI discrete-time system with an impulse response $\{h[n]\}$ shown below



- Its input-output relationship in the time-domain is given by the convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Eigen Function

- If the input is of the form

$$x[n] = e^{j\omega_o n}, \quad -\infty < n < \infty$$

then it follows that the output is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega_o (n-k)} = \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_o k} \right) e^{j\omega_o n}$$

- Let

$$H(e^{j\omega_o}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega_o k}$$

Eigen Function

- Then we can write

$$y[n] = H(e^{j\omega_o})e^{j\omega_o n}$$

- Thus for a complex exponential input signal $e^{j\omega_o n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega_o})$
- Thus $e^{j\omega_o n}$ is an eigen function of the system

The Frequency Response

For a **generic** ω,

- The **function** $H(e^{j\omega})$ is called the **frequency response** of the LTI discrete-time system
- $H(e^{j\omega})$ provides a frequency-domain description of the system
- $H(e^{j\omega})$ is precisely the DTFT of the impulse response $\{h[n]\}$ of the system

The Frequency Response

- $H(e^{j\omega})$, in general, is a complex function of ω with a period 2π
- It can be expressed in terms of its real and imaginary parts

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + j\,H_{im}(e^{j\omega})$$

or, in terms of its magnitude and phase,

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg H(e^{j\omega})$$

The Frequency Response

- The function $\left|H(e^{j\omega})\right|$ is called the **magnitude response** and the function $\theta(\omega)$ is called the **phase response** of the LTI discrete-time system
- Design specifications for the LTI discrete-time system, in many applications, are given in terms of the magnitude response or the phase response or both

The Frequency Response

- In some cases, the magnitude function is specified in **decibels** as

$$\mathcal{G}(\omega) = 20\log_{10}\left|H(e^{j\omega})\right| \quad dB$$

where $\mathcal{G}(\omega)$ is called the **gain function**

- The negative of the gain function

$$\mathcal{A}(\omega) = -\mathcal{G}(\omega)$$

is called the **attenuation** or **loss function**

The Frequency Response

- Note: Magnitude and phase functions are real functions of ω, whereas the frequency response is a complex function of ω
- If the impulse response $h[n]$ is real then it follows from Table 3.2 that the magnitude function is an even function of ω:

$$\left|H(e^{j\omega})\right| = \left|H(e^{-j\omega})\right|$$

and the phase function is an odd function of ω:

$$\theta(\omega) = -\theta(-\omega)$$

The Frequency Response

- Likewise, for a real impulse response $h[n]$, $H_{re}(e^{j\omega})$ is even and $H_{im}(e^{j\omega})$ is odd
- Example - Consider the M-point moving average filter with an impulse response given by

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Its frequency response is then given by

$$H(e^{j\omega}) = \frac{1}{M} \sum_{n=0}^{M-1} e^{-j\omega n}$$

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The Frequency Response

- Or, $H(e^{j\omega}) =$

$$= \frac{1}{M} \cdot \frac{1 - e^{-jM\omega}}{1 - e^{-j\omega}}$$

$$= \frac{1}{M} \cdot \frac{\sin(M\omega/2)}{\sin(\omega/2)} e^{-j(M-1)\omega/2}$$

The Frequency Response

- Thus, the magnitude response of the M-point moving average filter is given by

$$\left|H(e^{j\omega})\right| = \left|\frac{1}{M}\cdot\frac{\sin(M\omega/2)}{\sin(\omega/2)}\right|$$

and the phase response is given by

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi\sum_{k=0}^{\lfloor M/2 \rfloor}\mu\,(\omega - \frac{2\pi k}{M})$$

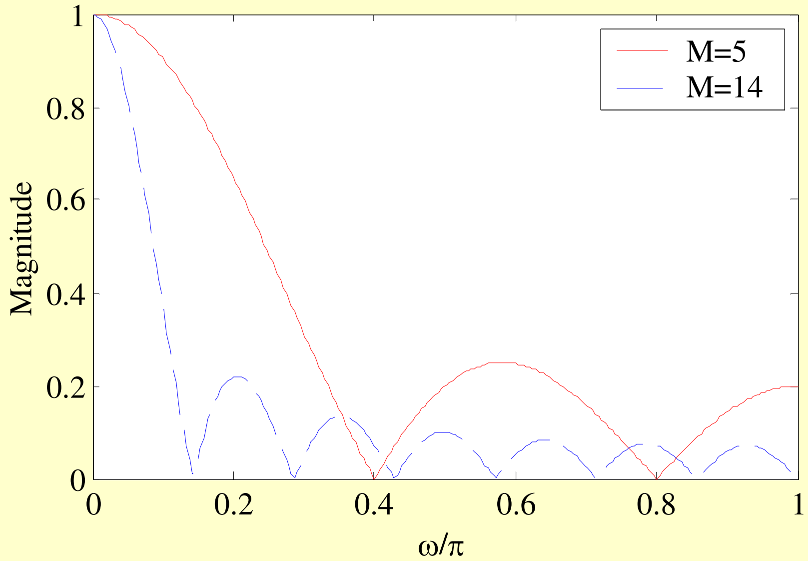
Linear component + regularly spaced jumps by π

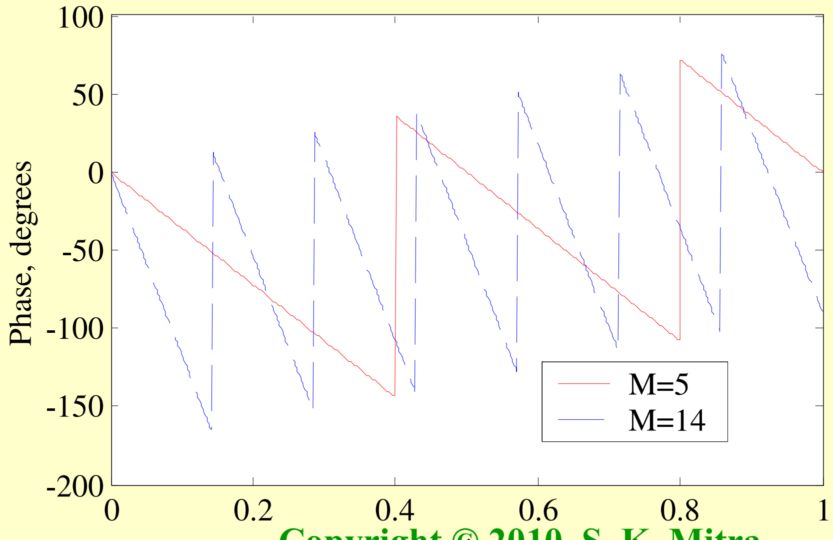
Frequency Response Computation Using MATLAB

- The function `freqz(h,1,w)` can be used to determine the values of the frequency response vector `h` at a set of given frequency points `w`
- From `h`, the real and imaginary parts can be computed using the functions `real` and `imag`, and the magnitude and phase functions using the functions `abs` and `angle`

Frequency Response Computation Using MATLAB

- Example - ~~Program 3_2.m~~ can be used to generate the magnitude and gain responses of an M-point moving average filter as shown below





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Frequency Response Computation Using MATLAB

- The phase response of a discrete-time system when determined by a computer may exhibit jumps by an amount 2π caused by the way the arctangent function is computed
- The phase response can be made a continuous function of ω by unwrapping the phase response across the jumps

Frequency Response Computation Using MATLAB

- To this end the function `unwrap` can be used, provided the computed phase is in radians
- The jumps by the amount of 2π should not be confused with the jumps caused by the zeros of the frequency response as indicated in the phase response of the moving average filter

Steady-State Response

- Note that the frequency response also determines the steady-state response of an LTI discrete-time system to a sinusoidal input
- Example - Determine the steady-state output $y[n]$ of a real coefficient LTI discrete-time system with a frequency response $H(e^{j\omega})$ for an input

$$x[n] = A\cos(\omega_o n + \phi), \quad -\infty < n < \infty$$

Steady-State Response

- We can express the input $x[n]$ as

$$x[n] = \tfrac{1}{2} A e^{j\phi} e^{j\omega_o n} + \tfrac{1}{2} A e^{-j\phi} e^{-j\omega_o n}$$
$$= g[n] + g^*[n]$$

where $g[n] = \tfrac{1}{2} A e^{j\phi} e^{j\omega_o n}$

- Now the output of the system for an input $e^{j\omega_o n}$ is simply

$$H(e^{j\omega_o}) e^{j\omega_o n}$$

Steady-State Response

- Because of linearity, the response $v[n]$ to an input $g[n]$ is given by

$$v[n] = \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n}$$

- Likewise, the output $v^*[n]$ to the input $g^*[n]$ is

$$v^*[n] = \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n}$$

Steady-State Response

- Combining the last two equations we get

$$y[n] = v[n] + v^*[n]$$

$$= \frac{1}{2} A e^{j\phi} H(e^{j\omega_o}) e^{j\omega_o n} + \frac{1}{2} A e^{-j\phi} H(e^{-j\omega_o}) e^{-j\omega_o n}$$

$$= \frac{1}{2} A \left| H(e^{j\omega_o}) \right| \left\{ e^{j\theta(\omega_o)} e^{j\phi} e^{j\omega_o n} + e^{-j\theta(\omega_o)} e^{-j\phi} e^{-j\omega_o n} \right\}$$

$$= A \left| H(e^{j\omega_o}) \right| \cos(\omega_o n + \theta(\omega_o) + \phi)$$

Steady-State Response

- Thus, the output $y[n]$ has the same sinusoidal waveform as the input with two differences:

(1) the amplitude is multiplied by $|H(e^{j\omega_o})|$, the value of the magnitude function at $\omega = \omega_o$

(2) the output has a **phase lag** relative to the input by an amount $\theta(\omega_o)$, the value of the phase function at $\omega = \omega_o$

Response to a Causal Exponential Sequence

- The expression for the steady-state response developed earlier assumes that the system is initially relaxed before the application of the input $x[n]$
- In practice, excitation $x[n]$ to a causal LTI discrete-time system is usually a right-sided sequence applied at some sample index $n = n_o$
- We develop the expression for the output for such an input

Response to a Causal Exponential Sequence

- Without any loss of generality, assume $x[n] = 0$ for $n < 0$
- From the input-output relation

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

we observe that for an input

$$x[n] = e^{j\omega n}\mu[n]$$

the output is given by

$$y[n] = \left(\sum_{k=0}^{n} h[k]e^{j\omega(n-k)} \right)\mu[n]$$

Response to a Causal Exponential Sequence

- Or, $y[n] = \left(\sum_{k=0}^{n} h[k] e^{-j\omega k} \right) e^{j\omega n} \mu[n]$
- The output for $n < 0$ is $y[n] = 0$
- The output for $n \geq 0$ is given by

$$y[n] = \left(\sum_{k=0}^{n} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$= \left(\sum_{k=0}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

Response to a Causal Exponential Sequence

- Or,

$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k} \right) e^{j\omega n}$$

- The first term on the RHS is the same as that obtained when the input is applied at $n = 0$ to an initially relaxed system and is the **steady-state response**:

$$y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$$

Response to a Causal Exponential Sequence

- The second term on the RHS is called the **transient response**:

$$y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

- To determine the effect of the above term on the total output response, we observe

$$\left|y_{tr}[n]\right| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega(k-n)}\right| \le \sum_{k=n+1}^{\infty}\left|h[k]\right| \le \sum_{k=0}^{\infty}\left|h[k]\right|$$

Response to a Causal Exponential Sequence

- For a causal, stable LTI IIR discrete-time system, $h[n]$ is absolutely summable
- As a result, the transient response $y_{tr}[n]$ is a bounded sequence
- Moreover, as $n \to \infty$,

$$\sum_{k=n+1}^{\infty} |h[k]| \to 0$$

and hence, the transient response decays to zero as n gets very large

Response to a Causal Exponential Sequence

- For a causal FIR LTI discrete-time system with an impulse response $h[n]$ of length $N+1$, $h[n]=0$ for $n>N$
- Hence, $y_{tr}[n]=0$ for $n>N-1$
- Here the output reaches the steady-state value $y_{sr}[n]=H(e^{j\omega})e^{j\omega n}$ at $n=N$

The Concept of Filtering

- One application of an LTI discrete-time system is to **pass** certain frequency components in an input sequence without any distortion (if possible) and to **block** other frequency components
- Such systems are called digital filters and one of the main subjects of discussion in this course

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The Concept of Filtering

- The key to the filtering process is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences

The Concept of Filtering

- Thus, by appropriately choosing the values of the magnitude function $\left|H(e^{j\omega})\right|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others

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The Concept of Filtering

- To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Concept of Filtering

- We apply an input

$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n, \qquad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

to this system

- Because of linearity, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right)$$

$$+ B\left|H(e^{j\omega_2})\right|\cos\left(\omega_2 n + \theta(\omega_2)\right)$$

The Concept of Filtering

- As

$$\left|H(e^{j\omega_1})\right| \cong 1, \quad \left|H(e^{j\omega_2})\right| \cong 0$$

the output reduces to

$$y[n] \cong A\left|H(e^{j\omega_1})\right| \cos\left(\omega_1 n + \theta(\omega_1)\right)$$

- Thus, the system acts like a **lowpass filter**
- In the following example, we consider the design of a very simple digital filter

The Concept of Filtering

- Example - The input consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample

frequency values should always include "π" !

- We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component
- For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:

$$h[0] = h[2] = \alpha, \qquad h[1] = \beta$$

The Concept of Filtering

- The convolution sum description of this filter is then given by

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha\, x[n] + \beta\, x[n-1] + \alpha\, x[n-2]$$

- $y[n]$ and $x[n]$ are, respectively, the output and the input sequences
- **Design Objective**: Choose suitable values of α and β so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample

The Concept of Filtering

- Now, the frequency response of the FIR filter is given by

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$

$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$

$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega}$$

$$= (2\alpha\cos\omega + \beta)e^{-j\omega}$$

The Concept of Filtering

- The magnitude and phase functions are

$$\left|H(e^{j\omega})\right| = 2\alpha\cos\omega + \beta$$

$$\theta(\omega) = -\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$\left|H(e^{j0.1})\right| = 2\alpha\cos(0.1) + \beta = 0$$

$$\left|H(e^{j0.4})\right| = 2\alpha\cos(0.4) + \beta = 1$$

- Solving the above two equations we get

$$\alpha = -6.76195$$
$$\beta = 13.456335$$

The Concept of Filtering

- Thus the output-input relation of the FIR filter is given by

$$y[n] = -6.76195(x[n] + x[n-2]) + 13.456335\, x[n-1]$$

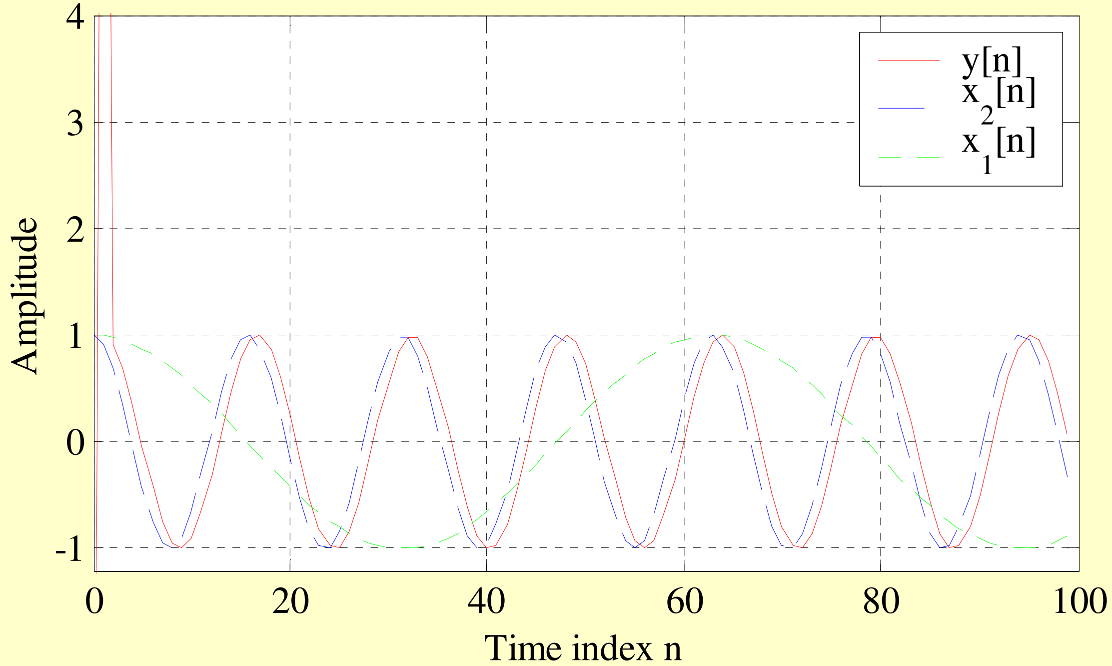
~~where the input is~~

$$\sout{x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]}$$

- Program 3_3.m can be used to verify the filtering action of the above system

The Concept of Filtering

- Figure below shows the plots generated by running this program



The Concept of Filtering

- The first seven samples of the output are shown below

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	−13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	−0.0291995	0.8918614	0.3623572
5	0.8775825	−0.4161468	0.4614357	−0.0292002
6	0.8253356	−0.7373937	0.0879419	−0.4161467

The Concept of Filtering

- From this table, it can be seen that, neglecting the least significant digit,

$$y[n] = \cos(0.4(n-1)) \qquad \text{for } n \geq 2$$

- Computation of the present value of the output requires the knowledge of the present and two previous input samples
- Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$

The Concept of Filtering

- Therefore, first two output samples constitute the transient part of the output
- Since the impulse response is of length 3, the steady-state is reached at $n = N = 2$
- Note also that the output is delayed version of the high-frequency component cos(0.4n) of the input, and the delay is one sample period

Phase Delay

- If the input $x[n]$ to an LTI system $H(e^{j\omega})$ is a sinusoidal signal of frequency ω_o:

$$x[n] = A\cos(\omega_o n + \phi), \quad -\infty < n < \infty$$

- Then, the output $y[n]$ is also a sinusoidal signal of the same frequency ω_o but lagging in phase by $\theta(\omega_o)$ radians:

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos(\omega_o n + \theta(\omega_o) + \phi),$$
$$-\infty < n < \infty$$

Phase Delay

- We can rewrite the output expression as

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos\left(\omega_o\left(n - \tau_p(\omega_o)\right) + \phi\right)$$

where

$$\tau_p(\omega_o) = -\frac{\theta(\omega_o)}{\omega_o}$$

is called the phase delay

- The minus sign in front indicates phase lag

Like in the case of a periodic sequence having period $N = (2\pi/\omega)r$ with $r > 1$ (slide 02.1-65)

Phase Delay

- Thus, the output $y[n]$ is a time-delayed version of the input $x[n]$
- In general, $y[n]$ will not be delayed replica of $x[n]$ unless the phase delay $\tau_p(\omega_o)$ is an integer

➡ Phase delay has a physical meaning only with respect to the underlying continuous-time functions associated with $y[n]$ and $x[n]$

Group Delay

- When the input is composed of many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays
- In this case, the signal delay is determined using the group delay defined by

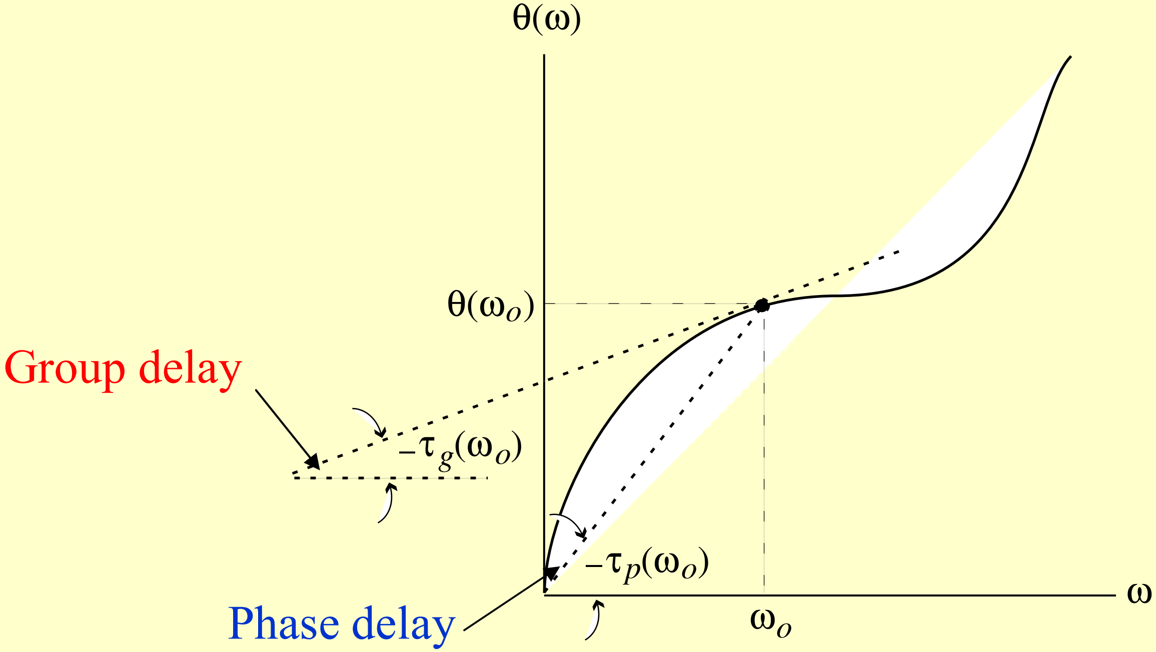
$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

Group Delay

- In defning the group delay, it is assumed that the phase function is unwrapped so that its derivatives exist
- Group delay also has a physical meaning only with respect to the underlying continuous-time functions associated with $y[n]$ and $x[n]$

Phase and Group Delays

- A graphical comparison of the two types of delays are indicated below



Phase and Group Delays

- Example - The phase function of the FIR filter $y[n] = \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$ is $\theta(\omega) = -\omega$ (see slide 39)
- Hence its group delay is given by $\tau_g(\omega) = 1$ verifying the result obtained earlier by simulation

Phase and Group Delays

- Example - For the M-point moving-average filter

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

the phase function is

$$\theta(\omega) = -\frac{(M-1)\omega}{2} + \pi \sum_{k=0}^{\lfloor M/2 \rfloor} \mu\left(\omega - \frac{2\pi k}{M}\right)$$

- Hence its group delay is

$$\tau_g(\omega) = \frac{M-1}{2}$$

Phase and Group Delays

- Physical significance of the two delays are better understood by examining the continuous-time case
- Consider an LTI continuous-time system with a frequency response

$$H_a(j\Omega) = |H_a(j\Omega)| e^{j\theta_a(\Omega)}$$

and excited by a narrow-band amplitude modulated continuous-time signal

$$x_a(t) = a(t)\cos(\Omega_c t)$$

Phase and Group Delays

- $a(t)$ is a lowpass modulating signal with a band-limited continuous-time Fourier transform given by

$$|A(j\Omega)| = 0, \quad |\Omega| > \Omega_o$$

and $\cos(\Omega_c t)$ is the high-frequency carrier signal

Phase and Group Delays

- We assume that in the frequency range $\Omega_c - \Omega_o < |\Omega| < \Omega_c + \Omega_o$ the frequency response of the continuous-time system has a constant magnitude and a linear phase:

$$|H_a(j\Omega)| = |H_a(j\Omega_c)|$$

$$\theta_a(\Omega) = \theta_a(\Omega_c) - (\Omega - \Omega_c)\left.\frac{d\theta_a(\Omega)}{d\Omega}\right|_{\Omega=\Omega_c}$$

$$= -\Omega_c \tau_p(\Omega_c) + (\Omega - \Omega_c)\tau_g(\Omega_c)$$

Phase and Group Delays

- Now, the CTFT of $x_a(t)$ is given by

$$X_a(j\Omega) = \frac{1}{2}\left(A(j[\Omega+\Omega_c]) + A(j[\Omega-\Omega_c])\right)$$

- Also, because of the band-limiting constraint $X_a(j\Omega)=0$ outside the frequency range $\Omega_c - \Omega_o < |\Omega| < \Omega_c + \Omega_o$

Phase and Group Delays

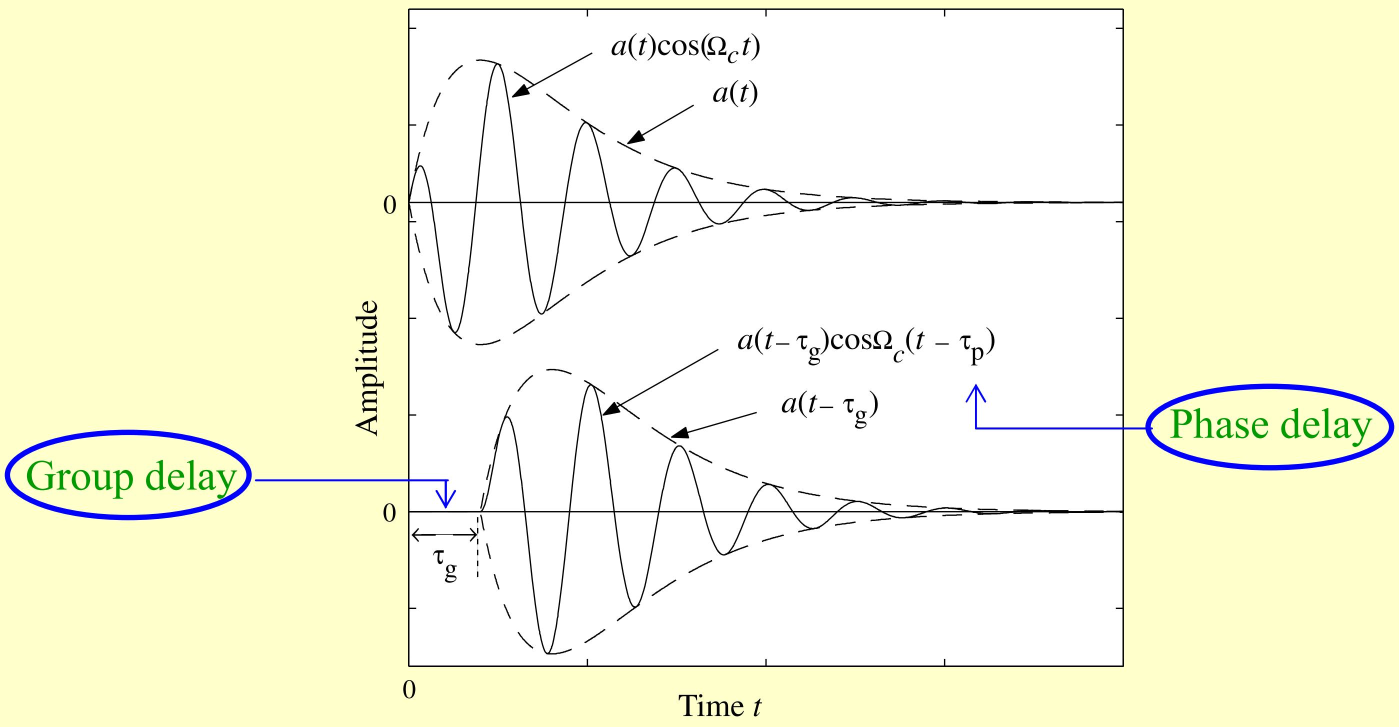
- As a result, the output response $y_a(t)$ of the LTI continuous-time system is given by

  $$y_a(t) = a\big(t - \tau_g(\Omega_c)\big)\cos\Omega_c\big(t - \tau_p(\Omega_c)\big)$$

  assuming $|H_a(j\Omega_c)| = 1$
- As can be seen from the above equation, the group delay $\tau_g(\Omega_c)$ is precisely the delay of the envelope $a(t)$ of the input signal $x_a(t)$, whereas, the phase delay $\tau_p(\Omega_c)$ is the delay of the carrier

Phase and Group Delays

- The figure below illustrates the effects of the two delays on an amplitude modulated sinusoidal signal



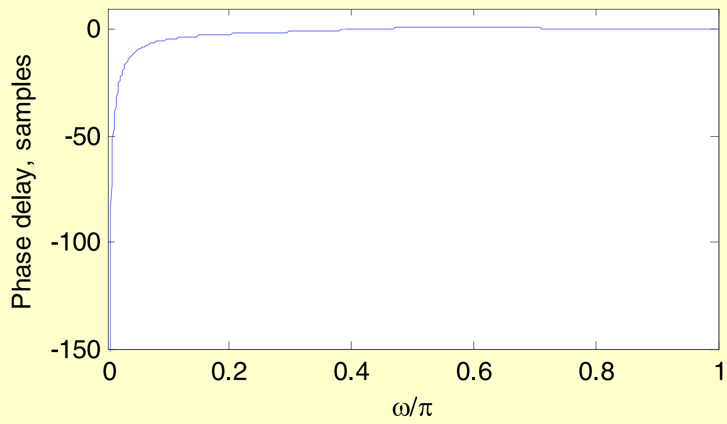
Phase and Group Delays

- The waveform of the underlying continuous-time output shows distortion when the group delay is not constant over the bandwidth of the modulated signal
- If the distortion is unacceptable, an allpass delay equalizer is usually cascaded with the LTI system so that the overall phase response is approximately linear over the frequency range of interest while keeping the magnitude response of the original LTI system unchanged

Phase Delay Computation Using MATLAB

- Phase delay can be computed using the function `phasedelay`
- Figure below shows the phase delay of the DTFT

$$H(e^{j\omega}) = \frac{0.1367(1 - e^{-j2\omega})}{1 - 0.5335e^{-j\omega} + 0.7265e^{-j2\omega}}$$



Group Delay Computation Using MATLAB

- Group delay can be computed using the function `grpdelay`
- Figure below shows the group delay of the DTFT

$$H(e^{j\omega}) = \frac{0.1367(1 - e^{-j2\omega})}{1 - 0.5335e^{-j\omega} + 0.7265e^{-j2\omega}}$$