z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

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z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  leads to the z-transform

- z-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of z-transform techniques permits simple algebraic manipulations

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z-Transform

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

where $z = \mathcal{R}e(z) + j\,\mathcal{I}m(z)$ is a complex variable

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z-Transform

- If we let $z = r\,e^{j\omega}$, then the z-transform reduces to

$$G(r\,e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}\,e^{-j\omega n}$$

- Thus $G(re^{j\omega})$ can be interpreted as the DTFT of the modified sequence $\{g[n]r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), z-transform reduces to its DTFT, provided the latter exists

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z-Transform

- The contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the **unit circle**
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set $\mathcal{R}$ of values of z for which its z-transform converges is called the **region of convergence** (ROC)

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z-Transform

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g[n]r^{-n} \right| < \infty$$

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z-Transform

- If $\sum_{n=-\infty}^{\infty} |g[n] r^{-n}| < \infty$ for $r = \mathcal{R}_{g-}$ and $r = \mathcal{R}_{g+}$ with $0 \le \mathcal{R}_{g-} < \mathcal{R}_{g+} < \infty$ then the sequence $g[n] r^{-n}$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |g[n] r^{-n}| < \infty$$

for all values of r in the range

$$0 \le \mathcal{R}_{g-} \le r \le \mathcal{R}_{g+} < \infty$$

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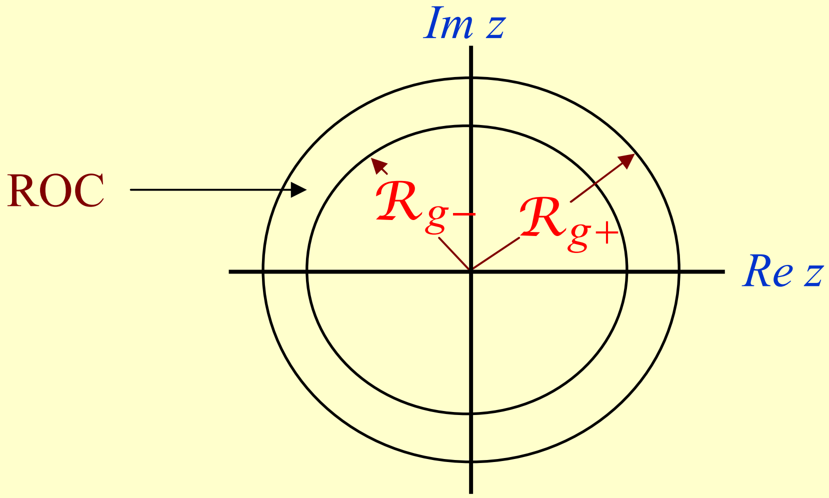
z-Transform

- The annular region defined by

$$0 \le \mathcal{R}_{g-} \le r \le \mathcal{R}_{g+} < \infty$$

is called the region of convergence (ROC) of $G(re^{j\omega}) = G(z)$



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z-Transform

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

- ROC is the annular region $|z| > |\alpha|$

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z-Transform

- Example - The z-transform $\mu(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

by setting $\alpha = 1$:

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z^{-1}| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

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z-Transform

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

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z-Transform

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^{n} z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^{m}$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^{m} = -\frac{\alpha^{-1} z}{1 - \alpha^{-1} z}$$

$$= \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region $|z| < |\alpha|$

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z-Transform

- Note: The *z*-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a *z*-transform is by specifying its ROC

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z-Transform

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the z-transform

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z-Transform

- Example - The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$$

which converges in the mean-square sense

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z-Transform

- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r

- Some commonly used z-transform pairs are listed on the next slide

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Table 6.1: Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$(r^n \cos\omega_o n)\mu[n]$	$\dfrac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n \sin\omega_o n)\mu[n]$	$\dfrac{(r\sin\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

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Rational z-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

note: this is the z-transform of the I/O relation of a recursive system

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Rational z-Transforms

- The **degree** of the numerator polynomial $P(z)$ is M and the **degree** of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

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Rational *z*-Transforms

- A rational *z*-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

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Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the **zeros** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the **poles** of $G(z)$

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Rational *z*-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M}(z-\xi_\ell)}{d_0 \prod_{\ell=1}^{N}(z-\lambda_\ell)}$$

- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

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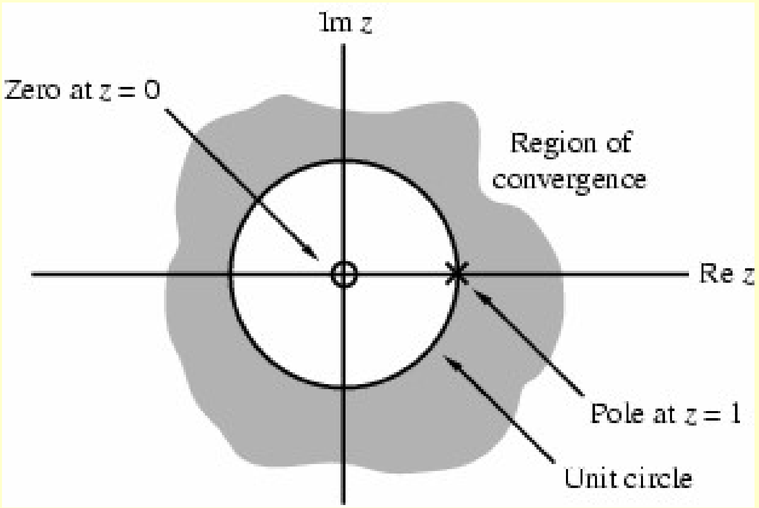
Rational *z*-Transforms

- Example - The *z*-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$



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Rational *z*-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1 - 2.4z^{-1} + 2.88z^{-2}}{1 - 0.8z^{-1} + 0.64z^{-2}}$$

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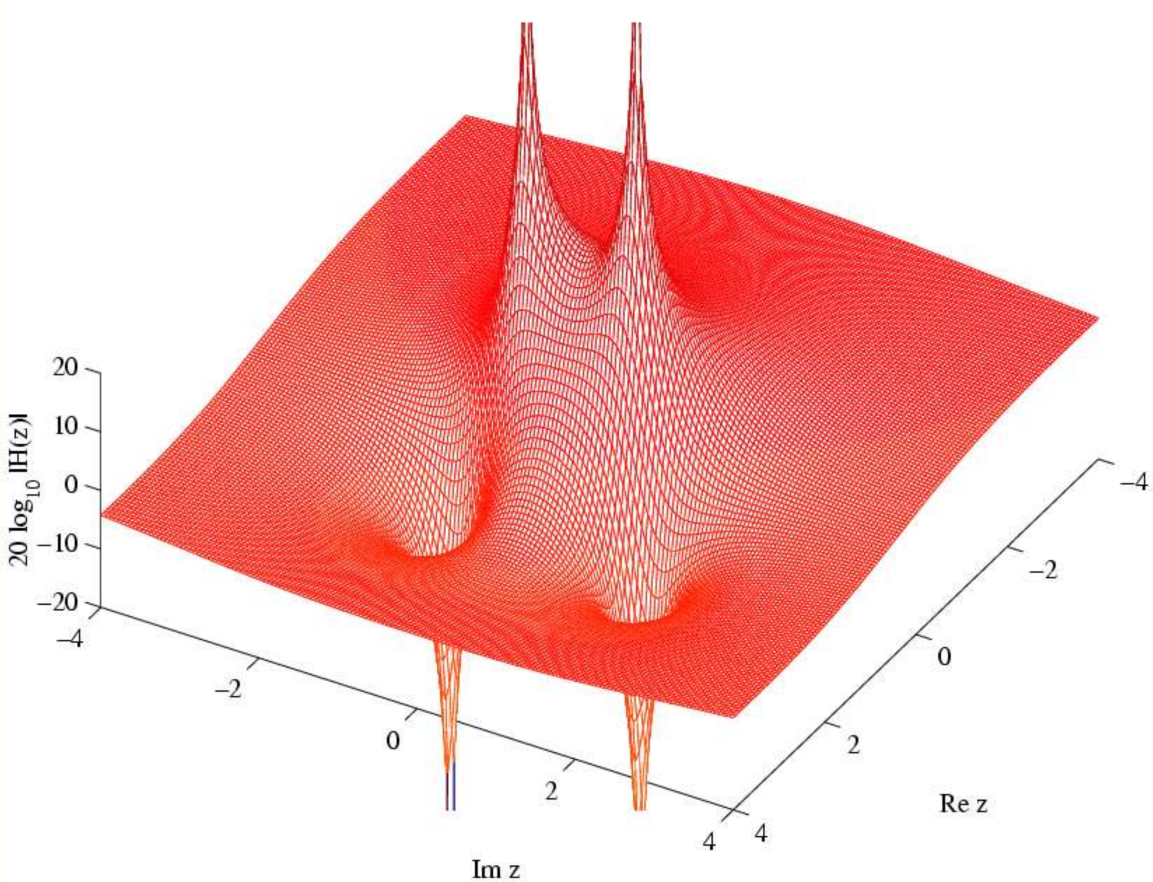
Rational *z*-Transforms



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Rational z-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at $z = 1.2 \pm j1.2$

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ROC of a Rational *z*-Transform

- ROC of a *z*-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its *z*-transform
- Hence, the *z*-transform must always be specified with its ROC

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ROC of a Rational z-Transform

- Moreover, if the ROC of a z-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the z-transform on the unit circle
- There is a relationship between the ROC of the z-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

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ROC of a Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$

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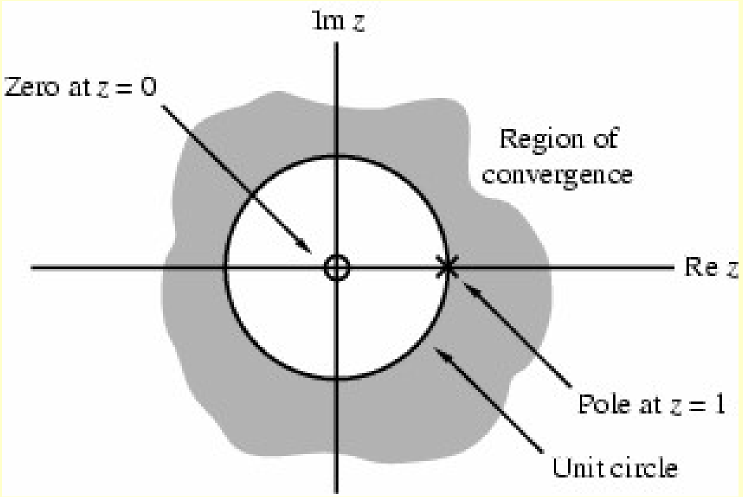
ROC of a Rational z-Transform

DTFT: no



- In this plot, the ROC, shown as the shaded area, is the region of the z-plane just outside the circle centered at the origin and going through the pole at $z = 1$

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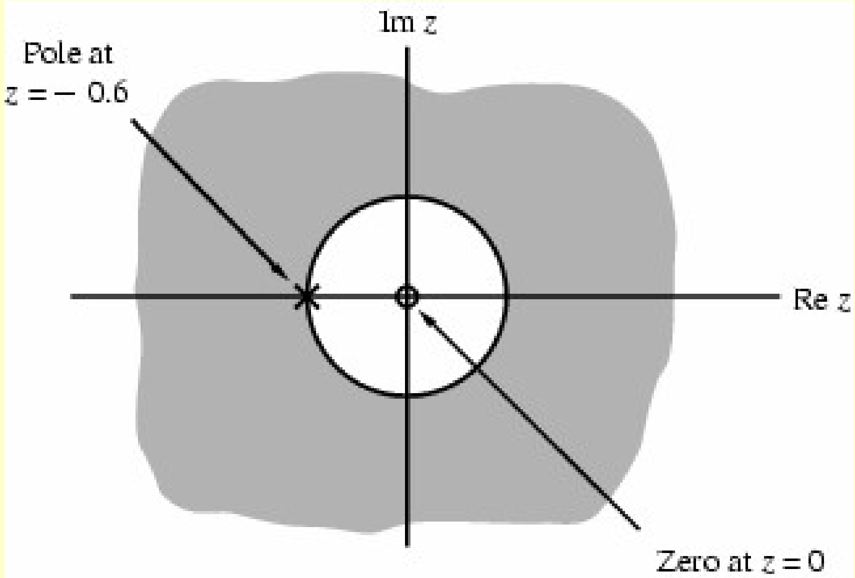
ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence $h[n] = (-0.6)^n \mu[n]$ is given by

$$H(z) = \frac{1}{1 + 0.6 z^{-1}},$$

$$|z| > 0.6$$



DTFT: yes

- Here the ROC is just outside the circle going through the point $z = -0.6$

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ROC of a Rational *z*-Transform

- A sequence can be one of the following types: finite-length, right-sided, left-sided and two-sided
- In general, the ROC depends on the type of the sequence of interest

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ROC of a Rational *z*-Transform

- Consider a finite-length sequence $g[n]$ defined for $-M \le n \le N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n] z^{-n} = \frac{\sum_{0}^{N+M} g[n-M] z^{N+M-n}}{z^{N}}$$

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ROC of a Rational z-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$
- As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$

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ROC of a Rational z-Transform

- A **right-sided sequence** with nonzero sample values for $n \geq 0$ is called a **causal sequence**
- Consider a causal sequence $u_1[n]$
- Its z-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

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ROC of a Rational z-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a z-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

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ROC of a Rational z-Transform

- A **left-sided sequence** with nonzero sample values for $n \leq 0$ is called a **anticausal sequence**
- Consider an anticausal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

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ROC of a Rational z-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \leq N$ with N nonnegative has a z-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

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ROC of a Rational z-Transform

- The z-transform of a two-sided sequence $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n]z^{-n} = \sum_{n=0}^{\infty} w[n]z^{-n} + \sum_{n=-\infty}^{-1} w[n]z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n]z^{-n}$, can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

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ROC of a Rational *z*-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n]z^{-n}$, can be interpreted as the z-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the z-transform does not exist

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ROC of a Rational z-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

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ROC of a Rational *z*-Transform

- There is no overlap between these two regions
- Hence, the *z*-transform of $u[n] = \alpha^n$ does not exist

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ROC of a Rational *z*-Transform

- The ROC is bounded on the outside by the pole with the smallest magnitude that contributes for $n < 0$ and on the inside by the pole with the largest magnitude that contributes for $n \geq 0$
- There are three possible ROCs of a rational *z*-transform with poles at $z = \alpha$ and $z = \beta$ ($|\alpha| < |\beta|$)

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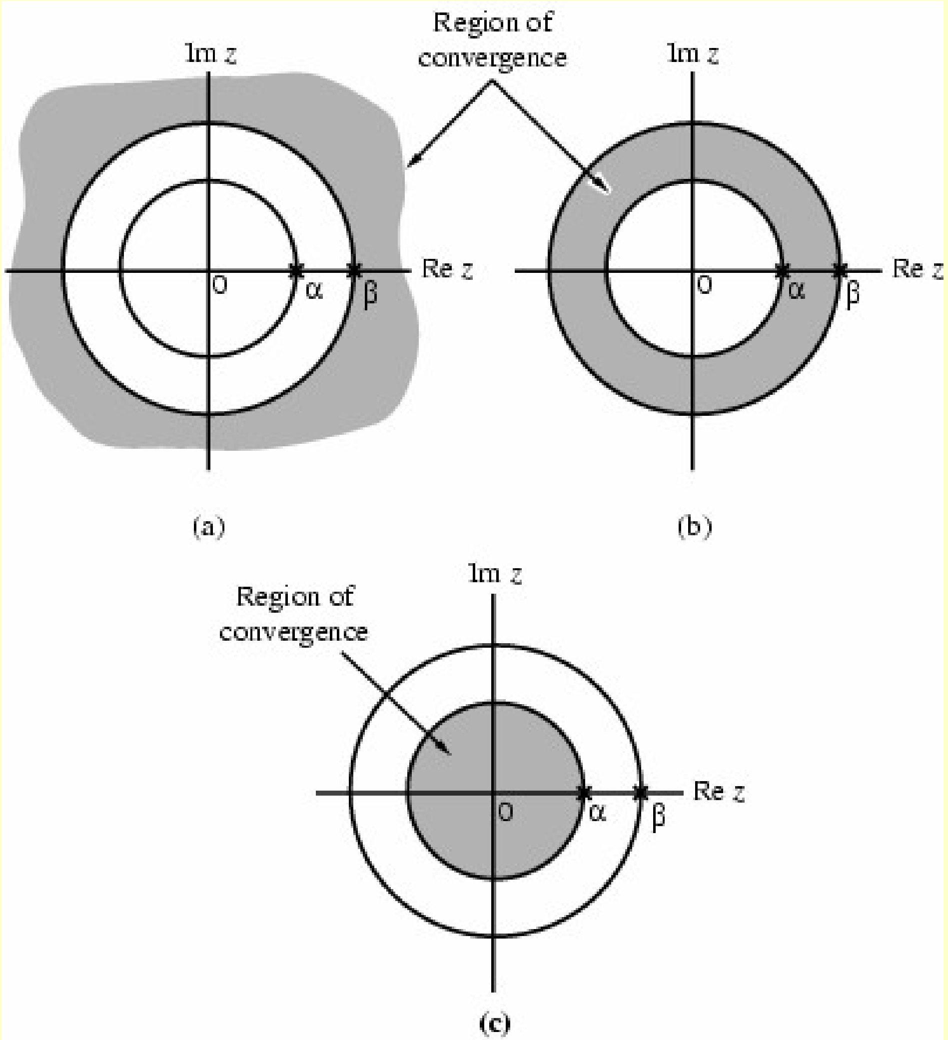
ROC of a Rational *z*-Transform



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ROC of a Rational z-Transform

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has $R+1$ ROCs
- Thus, there are $R+1$ distinct sequences with the same z-transform
- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform

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ROC of a Rational z-Transform

- The ROC of a rational z-transform can be easily determined using MATLAB

```
[z,p,k] = tf2zp(num,den)
```

determines the zeros, poles, and the gain constant of a rational z-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

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ROC of a Rational *z*-Transform

- `[num,den] = zp2tf(z,p,k)` implements the reverse process
- The factored form of the *z*-transform can be obtained using `sos = zp2sos(z,p,k)`
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix `sos`

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ROC of a Rational z-Transform

$$sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

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ROC of a Rational *z*-Transform

- The pole-zero plot is determined using the function `zplane`
- The *z*-transform can be either described in terms of its `zeros` and `poles`:

`zplane(zeros,poles)` (if column vectors)

- or, it can be described in terms of its numerator and denominator coefficients:

`zplane(num,den)` (if row vectors)

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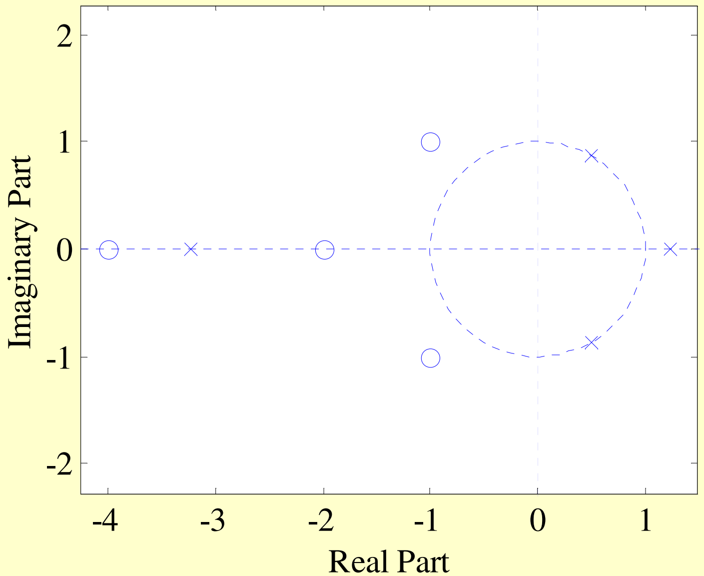
ROC of a Rational z-Transform

- Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below



× − pole
o − zero

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Inverse z-Transform

- **General Expression**: Recall that, for $z = r\,e^{j\omega}$, the z-transform $G(z)$ given by

  $$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

  is merely the DTFT of the modified sequence $g[n] r^{-n}$

- Accordingly, the inverse DTFT is thus given by

  $$g[n] r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(re^{j\omega}) e^{j\omega n} d\omega$$

Inverse z-Transform

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

$$g[n] = \frac{1}{2\pi j} \oint_{C'} G(z)\, z^{n-1} dz$$

where C' is a counterclockwise contour of integration defined by $|z| = r$

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Inverse z-Transform

- But the integral remains unchanged when C' is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the Cauchy's residue theorem resulting in

$$g[n] = \sum \begin{bmatrix} \text{residues of } G(z)z^{n-1} \\ \text{at the poles inside } C \end{bmatrix}$$

- The above equation needs to be evaluated at all values of n and is not pursued here

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Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has an ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

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Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

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Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a **proper fraction**
- To develop the proper fraction part $P_1(z)/D(z)$ from $G(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reverse order until the remainder polynomial $P_1(z)$ is of lower degree than that of the denominator $D(z)$ (Method of Residues)

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Partial-Fraction Expansion Using MATLAB

- `[r,p,k]= residuez(num,den)` develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors `num` and `den`
- Vector `r` contains the residues
- Vector `p` contains the poles
- Vector `k` contains the constants η_ℓ

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Partial-Fraction Expansion Using MATLAB

- `[num,den]=residuez(r,p,k)` converts a *z*-transform expressed in a partial-fraction expansion form to its rational form

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Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division

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Inverse z-Transform via Long Division

Transfer function of an IIR filter

- Example - Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

and of its FIR approximation

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- As a result

$$\{h[n]\} = \{\underset{\uparrow}{1} \quad 1.6 \quad -0.52 \quad 0.4 \quad -0.2224 \quad \cdots\}, \quad n \ge 0$$

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Inverse z-Transform Using MATLAB

- The function `impz` can be used to find the inverse of a rational *z*-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

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Table 6.2: z-Transform Theorems

Theorems	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\|\alpha\|\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation		$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$	

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

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