DTFT Theorems

DTFT of a recursively defined sequence

- Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.3, the DTFT of $\delta[n]$ is 1
- Using the time-shifting theorem of the DTFT given in Table 3.4 we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Theorems

- Using the linearity theorem of Table 3.4 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Energy Density Spectrum

- The total energy of a finite-energy sequence $g[n]$ is given by

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2$$

- From Parseval's theorem given in Table 3.4 we observe that

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} |g[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \, d\omega$$

Energy Density Spectrum

- The quantity

$$S_{gg}(\omega) = \left| G(e^{j\omega}) \right|^2$$

is called the **energy density spectrum**

- The area under this curve in the range $-\pi < \omega \leq \pi$ divided by 2π is the energy of the sequence

Band-limited Discrete-time Signals

- Since the spectrum of a discrete-time signal is a periodic function of ω with a period 2π, a full-band signal has a spectrum occupying the frequency range $-\pi < \omega \leq \pi$
- A band-limited discrete-time signal has a spectrum that is limited to a portion of the frequency range $-\pi < \omega \leq \pi$

Band-limited Discrete-time Signals

- An ideal band-limited signal has a spectrum that is zero outside a frequency range $0 < \omega_a \le |\omega| \le \omega_b < \pi$, that is

$$X(e^{j\omega}) = \begin{cases} 0, & 0 \le |\omega| < \omega_a \\ 0, & \omega_b < |\omega| < \pi \end{cases}$$

$$X(e^{j\omega}) \ne 0, \qquad \omega_a \le |\omega| \le \omega_b$$

- An ideal band-limited discrete-time signal cannot be generated in practice

Band-limited Discrete-time Signals

- A classification of a band-limited discrete-time signal is based on the frequency range where most of the signal's energy is concentrated

should be: "low-frequency"

- A lowpass discrete-time real signal has a spectrum occupying the frequency range $0 < |\omega| \le \omega_p < \pi$ and has a bandwidth of ω_p

Band-limited Discrete-time Signals

- A highpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_p \le |\omega| < \pi$ and has a bandwidth of $\pi - \omega_p$
- A bandpass discrete-time real signal has a spectrum occupying the frequency range $0 < \omega_L \le |\omega| \le \omega_H < \pi$ and has a bandwidth of $\omega_H - \omega_L$

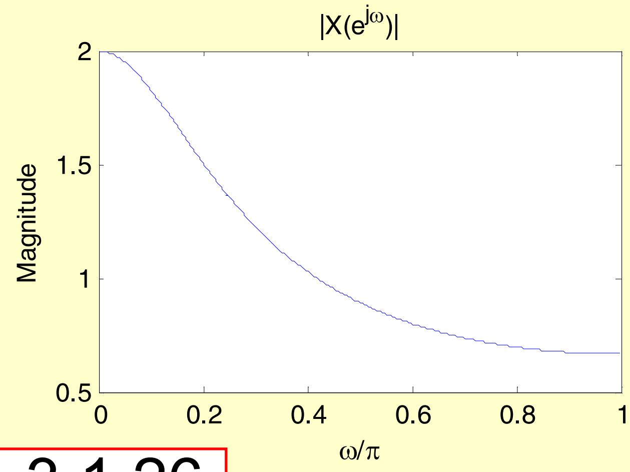
Band-limited Discrete-time Signals

- Example – Consider the sequence

$$x[n] = (0.5)^n \mu[n]$$

- Its DTFT is given below on the left along with its magnitude spectrum shown below on the right

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



see slide 3.1.26

Band-limited Discrete-time Signals

- It can be shown that 80% of the energy of this lowpass signal is contained in the frequency range $0 \le |\omega| \le 0.5081\pi$
- Hence, we can define the 80% bandwidth to be 0.5081π radians

Energy Density Spectrum

- Example - Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{LP}(e^{j\omega})|^2 d\omega$$

where

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

Energy Density Spectrum

- Therefore

$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n] \right|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

- Hence, $h_{LP}[n]$ is a finite-energy lowpass sequence

(already seen in 3.1.41)

rem.: sequence is not absolutely summable

DTFT Computation Using MATLAB

Can we do this?

- The function `freqz` can be used to compute the values of the DTFT of a sequence, described as a rational function in in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + ... + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + ... + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_\ell$ ...dense enough to look continuous

DTFT Computation Using MATLAB

- For example, the statement

```
H = freqz(num,den,w)
```

returns the frequency response values as a vector `H` of a DTFT defined in terms of the vectors `num` and `den` containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector `w`

see Matlab example E03_1

DTFT Computation Using MATLAB

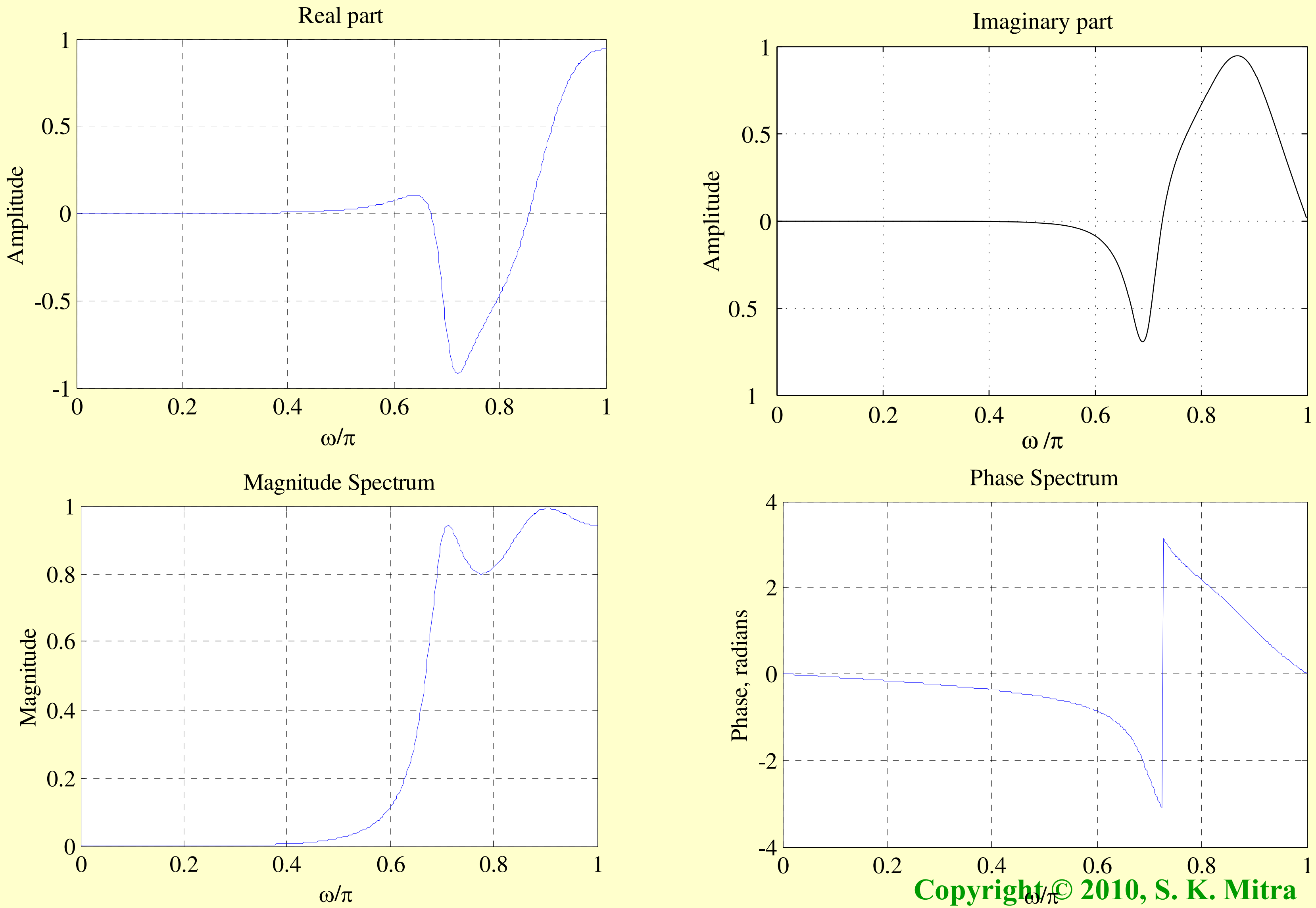
- Example - Plots of the real and imaginary parts, and the magnitude and phase of the DTFT as a function of the normalized angular frequency variable ω/π

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

are shown on the next slide

DTFT Computation Using MATLAB

MATLAB



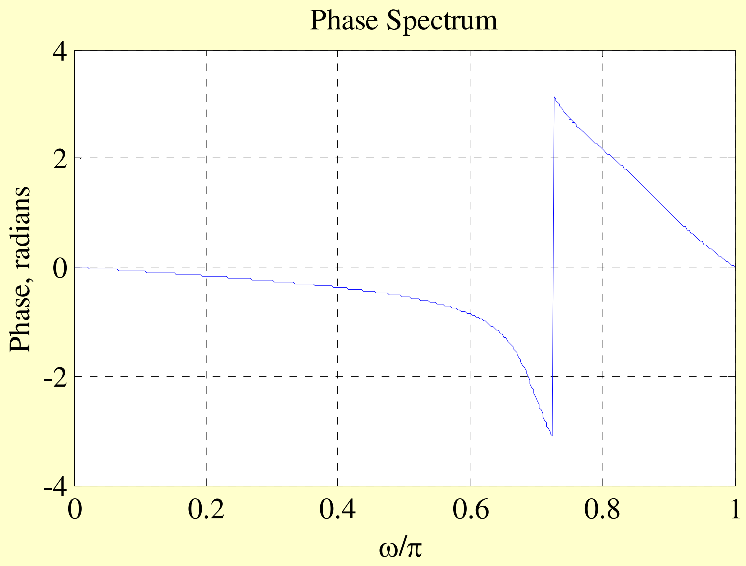
The Unwrapped Phase Function

- In numerical computation, when the computed phase function is outside the range $[-\pi, \pi]$, the phase is computed modulo 2π, to bring the computed value to this range
- Thus. the phase functions of some sequences exhibit discontinuities of 2π radians in the plot

The Unwrapped Phase Function

- For example, there is a discontinuity of 2π at $\omega = 0.72$ in the phase response below

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

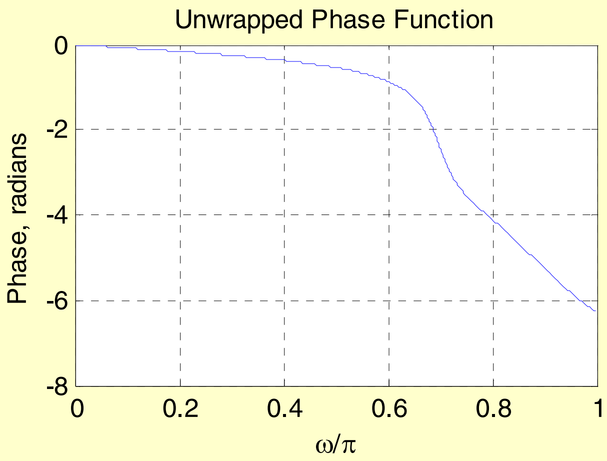


The Unwrapped Phase Function

- In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π
- Process of discontinuity removal is called unwrapping the phase
- The unwrapped phase function will be denoted as $\theta_c(\omega)$

The Unwrapped Phase Function

- In MATLAB, the unwrapping can be implemented using the M-file `unwrap`
- The unwrapped phase function of the DTFT of previous page is shown below



MATLAB

Linear Convolution Using DTFT

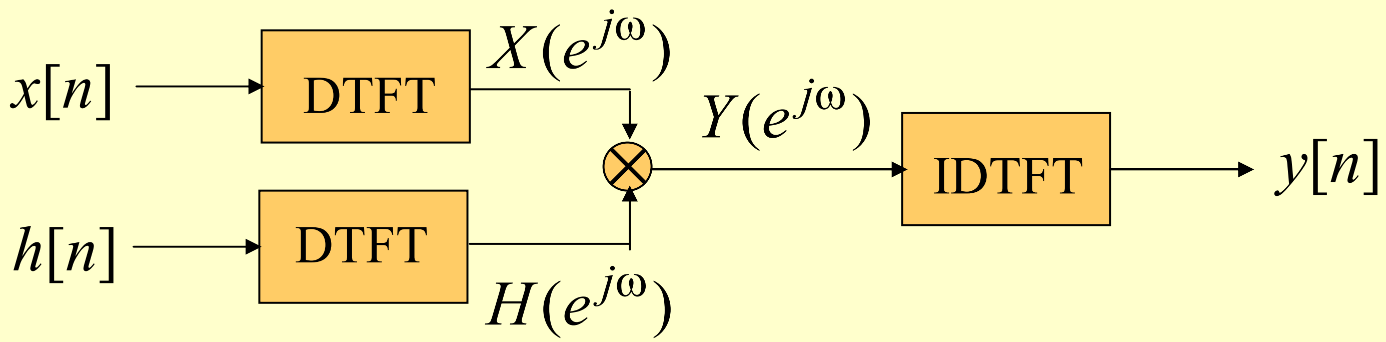
- An important property of the DTFT is given by the convolution theorem in Table 3.4
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Linear Convolution Using DTFT

- 1) Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
- 2) Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



Orthogonal Transforms

- Let $x[n], 0 \le n \le N-1$, denote a length-N time-domain sequence
- Let $X[k], 0 \le k \le N-1$, denote the coefficients of the N-point orthogonal transform of $x[n]$

1

Orthogonal Transforms

- A general form of the orthogonal transform pair is of the form

$$\mathcal{X}[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k,n], \quad 0 \le k \le N-1$$

Analysis equation

$$x[n] = 1/N \sum_{k=0}^{N-1} \mathcal{X}[k]\psi[k,n], \quad 0 \le n \le N-1$$

Synthesis equation

- $\psi[k,n]$, called the basis sequences, are also length-N sequences

Orthogonal Transforms

- In the class of transforms to be considered in this course, the basis sequences satisfy the condition

$$\frac{1}{N}\sum_{n=0}^{N-1}\psi[k,n]\psi^*[\ell,n]=\begin{cases}1, & \ell=k\\ 0, & \ell\neq k\end{cases}$$

- Basis sequences satisfying the above condition are said to be orthogonal to each other

Orthogonal Transforms

- To verify the inverse transform expression

$$x[\,n\,] = 1/N \sum_{k=0}^{N-1} \mathcal{X}[k]\psi[k,n], \quad 0 \le n \le N-1$$

we substitute it into

$$\mathcal{X}[k] = \sum_{n=0}^{N-1} x[n]\psi^*[k,n], \quad 0 \le k \le N-1$$

Orthogonal Transforms

- The substitution yields

$$\sum_{n=0}^{N-1} x[n]\psi^*[\ell,n] = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]\psi[k,n]\right)\psi^*[\ell,n]$$

$$= \sum_{k=0}^{N-1} X[k]\left(\frac{1}{N}\sum_{n=0}^{N-1}\psi[k,n]\psi^*[\ell,n]\right) = X[\ell]$$

Orthogonal Transforms

- Energy Preservation Property-

$$\sum_{n=0}^{N-1} \left| x[n] \right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left| X[k] \right|^2$$

- An important consequence of the orthogonality of the basis sequences
- More commonly known as the Parseval's theorem

Discrete Fourier Transform

- Definition - Given a length-N sequence $x[n]$, defined for $0 \leq n \leq N-1$, and its DTFT $X(e^{j\omega})$ by uniformly sampling $X(e^{j\omega})$ on the ω-axis between $0 \leq \omega < 2\pi$ at $\omega_k = 2\pi k/N$, $0 \leq k \leq N-1$ we get a sequence X[k]:
- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad 0 \leq k \leq N-1$$

Discrete Fourier Transform

- Note: $X[k]$ is also a length-N sequence in the frequency domain
- The sequence $X[k]$ is called the **discrete Fourier transform** (DFT) of the sequence $x[n]$
- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \; 0 \le k \le N-1}$$

Discrete Fourier Transform

- The **inverse discrete Fourier transform** (IDFT) is given by

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1}$$

- To verify the above expression we multiply both sides of the above equation by $W_N^{\ell n}$ and sum the result from $n = 0$ to $n = N-1$

Discrete Fourier Transform

resulting in

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right) W_N^{\ell n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} X[k] W_N^{-(k-\ell)n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} \right)$$

Discrete Fourier Transform

- From the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, \quad \text{otherwise} \end{cases}$$

it follows then that the only non-zero term in

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n}$$

is obtained when $k = \ell$ as $0 \leq k, \ell \leq N-1$

Discrete Fourier Transform

- Hence

$$\sum_{n=0}^{N-1} x[n] W_N^{\ell n} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] \left(\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} \right)$$

$$= \frac{1}{N} \cdot X[\ell] \cdot N = X[\ell]$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \le n \le m-1,\ m+1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$$0 \le k \le N-1$$

Discrete Fourier Transform

- Example - Consider the length-N sequence defined for $0 \le n \le N-1$

$$g[n] = \cos(2\pi rn/N), \quad 0 < r \le N-1$$

- Using a trigonometric identity we can write

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

Discrete Fourier Transform

- The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \leq k \leq N-1$$

Discrete Fourier Transform

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-\ell)n} = \begin{cases} N, \text{ for } k-\ell = rN, & r \text{ an integer} \\ 0, \quad \text{otherwise} \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N-r \\ 0, & \text{otherwise} \end{cases}$$

MATLAB

$$0 \le k \le N-1$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

where

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \cdots\cdots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots\cdots & x[N-1] \end{bmatrix}^T$$

Matrix Relations

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

Matrix Relations

- Likewise, the IDFT relation given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

Matrix Relations

where

$$\mathbf{D}_N^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \boxed{(1/N)}$$

- Note:

$$\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^{*}$$

(unitary matrix)

DFT Computation Using MATLAB

Can we do this?

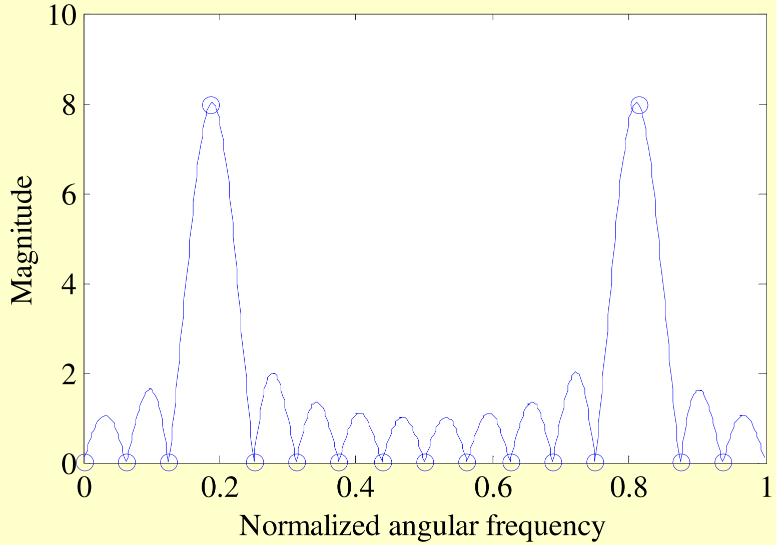
- The functions to compute the DFT and the IDFT are `fft` and `ifft`
- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation
- Programs 5_1.m and 5_2.m illustrate the use of these functions

DFT Computation Using MATLAB

- Example - Program 5_3.m can be used to compute the DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

as shown below



○ indicates DFT samples

Sampling the DTFT

of any given length M

- Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k/N,\ 0 \leq k \leq N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$

Sampling the DTFT

- Now $$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$$

- Thus $$Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$$
$$= \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\ell/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}$$

- An IDFT of $Y[k]$ yields
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}$$

Sampling the DTFT

- i.e. $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{\ell=-\infty}^{\infty} x[\ell]W_N^{k\ell}\,W_N^{-kn}$$
$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\ell)}\right]$$

- Making use of the identity
$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-r)} = \begin{cases} 1, & \text{for } r = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Sampling the DTFT

we arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \;\; 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

Sampling the DTFT

- To apply

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \;\; 0 \leq n \leq N-1$$

to finite-length sequences, we assume that the samples outside the specified range are zeros

- Thus if $x[n]$ is a length-M sequence with $M \leq N$, then $y[n] = x[n]$ for $0 \leq n \leq N-1$

Sampling the DTFT

- If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$
- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

Sampling the DTFT

$$y[n] = x[n] + x[n+4] + x[n-4]\ ,\ 0 \le n \le 3$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

⇒ $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

Sampling the DTFT

$x(n)$ → DTFT → $X(\exp(j\omega))$ → sample → $Y(k)$ → IDFT → $y(n)$

Length(x) = M; Length(y) = N; $y(n) = \Sigma_m x(n+mN)$

e.g. $M = 5$; $x = \{3\ 4\ 5\ 6\ 7\}$ (with implicit zero-padding)

- If $N = 5$: $y = \underbrace{\{3\ 4\ 5\ 6\ 7\}}_{m=0} + \underbrace{\{0\ 0\ 0\ 0\ 0\}}_{m=1} + \underbrace{\{0\ 0\ 0\ 0\ 0\}}_{m=2} + \ldots$

  → Ok

- If $N = 7$: $y = \{3\ 4\ 5\ 6\ 7\ 0\ 0\} + \{0\ 0\ 0\ 0\ 0\ 0\ 0\} + \{0\ 0\ 0\ 0\ 0\ 0\ 0\} + \ldots$

  → Ok

- If $N = 3$: $y = \{3\ 4\ 5\} + \{6\ 7\ 0\} + \{0\ 0\ 0\} + \ldots = \{9\ \ 11\ \ 5\}$

  → Aliasing

Approximated DTFT by zero-padding *x(n)*

Case $N=7$ above suggests that a higher sample rate DFT can be obtained by transforming a zero-padded $x(n)$

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DTFT from DFT by Interpolation

The N-point DFT of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly-spaced frequency points

$$\omega = \omega_k = 2\pi k/N, \quad 0 \le k \le N-1$$

Vice-versa, given the DFT of a finite-length sequence, an approximated DTFT of the sequence can be obtained by interpolation

Numerical Computation of the DTFT Using the DFT

- A practical approach to the numerical approximation of the DTFT of a finite-length sequence
- Let $X(e^{j\omega})$ be the DTFT of a length-N sequence $x[n]$
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k / M$, $0 \leq k \leq M-1$, where $M >> N$:

Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n] e^{-j2\pi kn/M}$$

Numerical Computation of the DTFT Using the DFT

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$
- The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if M is an integer power of 2
- The function `freqz` employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function in $e^{-j\omega}$

MATLAB

Periodicity of DFT and IDFT

Due to the periodic nature of the complex exponential, both the DFT and IDFT are periodic sequences:

$$X_{k+N} \triangleq \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi}{N}(k+N)n} = \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi}{N}kn} \underbrace{e^{-i2\pi n}}_{1} = \sum_{n=0}^{N-1} x_n e^{-\frac{i2\pi}{N}kn} = X_k$$

$$x(n+N) = (1/N)\Sigma_k X(k) \exp(i\, 2\pi\, k\, (n+N) / N) = \ldots = x(n)$$

This is the reason of the circular time-shifting property of the DFT

(later)

see FourierOverviewTable.pdf