Discrete-Time Signals: Time-Domain Representation

- Signals represented as sequences of numbers, called **samples**
- Sample value of a typical signal or sequence denoted as $x[n]$ with n being an integer in the range $-\infty \leq n \leq \infty$
- $x[n]$ defined only for integer values of n and undefined for noninteger values of n
- Discrete-time signal represented by $\{x[n]\}$

Discrete-Time Signals: Time-Domain Representation

- Discrete-time signal may also be written as a sequence of numbers inside braces:

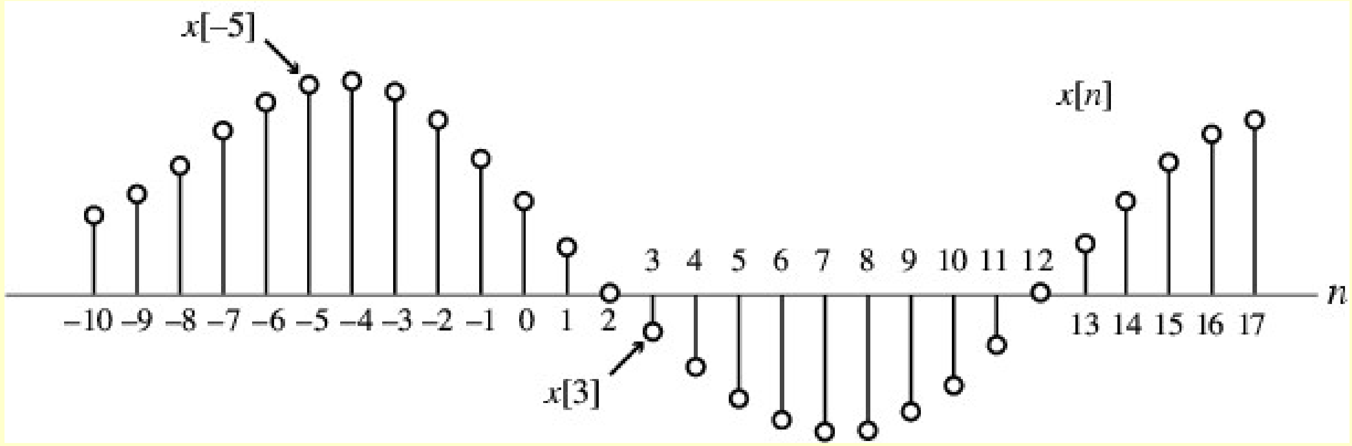
$$\{x[n]\} = \{\ldots, -0.2, \underset{\uparrow}{2.2}, 1.1, 0.2, -3.7, 2.9, \ldots\}$$

- In the above, $x[-1] = -0.2$, $x[0] = 2.2$, $x[1] = 1.1$, etc.

- The arrow is placed under the sample at time index $n = 0$

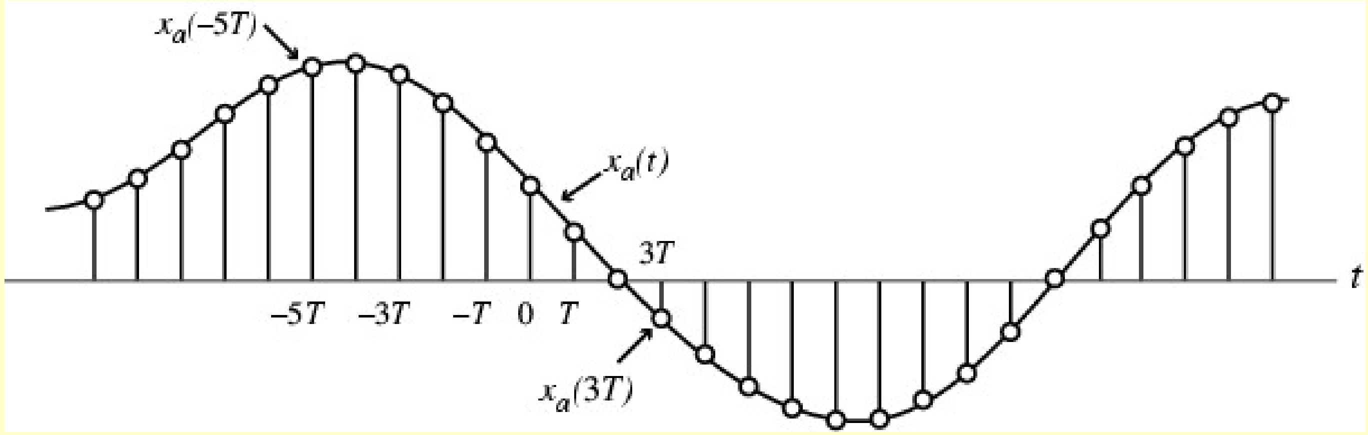
Discrete-Time Signals: Time-Domain Representation

- Stem plot: Graphical representation of a discrete-time signal with real-valued samples



Discrete-Time Signals: Time-Domain Representation

- In some applications, a discrete-time sequence $\{x[n]\}$ may be generated by periodically sampling a continuous-time signal $x_a(t)$ at uniform intervals of time



Discrete-Time Signals: Time-Domain Representation

- Here, n-th sample is given by

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT),\ n = \ldots,-2,-1,0,1,\ldots$$

disregarding quantization errors

The spacing T between two consecutive samples is called the **sampling interval** or **sampling period**

- Reciprocal of sampling interval T, denoted as F_T, is called the **sampling frequency**:

$$F_T = \frac{1}{T}$$

F_S in Matlab

Discrete-Time Signals: Time-Domain Representation

- Unit of sampling frequency is cycles per second, or **hertz** (Hz), if T is in seconds
- Whether or not the sequence $\{x[n]\}$ has been obtained by sampling, the quantity $x[n]$ is called the n**-th sample** of the sequence
- $\{x[n]\}$ is a **real sequence**, if the n-th sample $x[n]$ is real for all values of n
- Otherwise, $\{x[n]\}$ is a **complex sequence**

Discrete-Time Signals: Time-Domain Representation

- A complex sequence $\{x[n]\}$ can be written as $\{x[n]\} = \{x_{re}[n]\} + j\{x_{im}[n]\}$ where $x_{re}[n]$ and $x_{im}[n]$ are the real and imaginary parts of $x[n]$
- The complex conjugate sequence of $\{x[n]\}$ is given by $\{x^*[n]\} = \{x_{re}[n]\} - j\{x_{im}[n]\}$
- Often the braces are ignored to denote a sequence if there is no ambiguity

Discrete-Time Signals: Time-Domain Representation

- Example - $\{x[n]\} = \{\cos 0.25n\}$ is a real sequence
- $\{y[n]\} = \{e^{j0.3n}\}$ is a complex sequence
- We can write

$$\{y[n]\} = \{\cos 0.3n + j \sin 0.3n\}$$

$$= \{\cos 0.3n\} + j\{\sin 0.3n\}$$

where $\{y_{re}[n]\} = \{\cos 0.3n\}$

$$\{y_{im}[n]\} = \{\sin 0.3n\}$$

Discrete-Time Signals: Time-Domain Representation

- Example -

  $$\{w[n]\} = \{\cos 0.3n\} - j\{\sin 0.3n\} = \{e^{-j0.3n}\}$$

  is the complex conjugate sequence of $\{y[n]\}$
- That is,

  $$\{w[n]\} = \{y^*[n]\}$$

Discrete-Time Signals: Time-Domain Representation

- Two types of discrete-time signals:

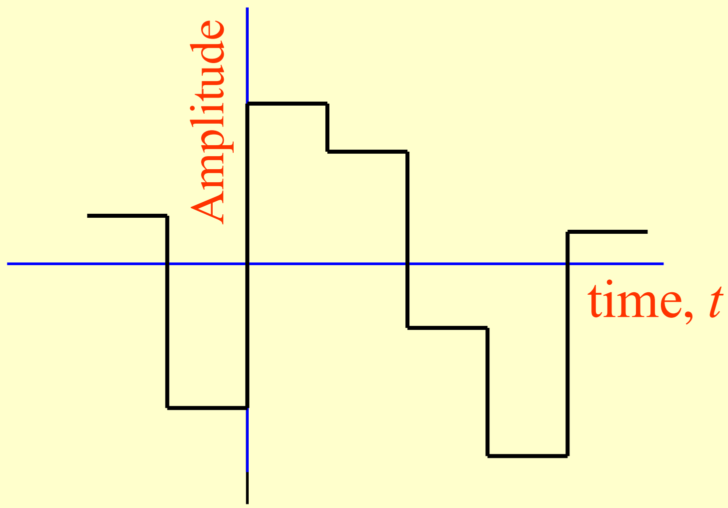
  - **Sampled-data signals** in which samples are continuous-valued

  - **Digital signals** in which samples are discrete-valued

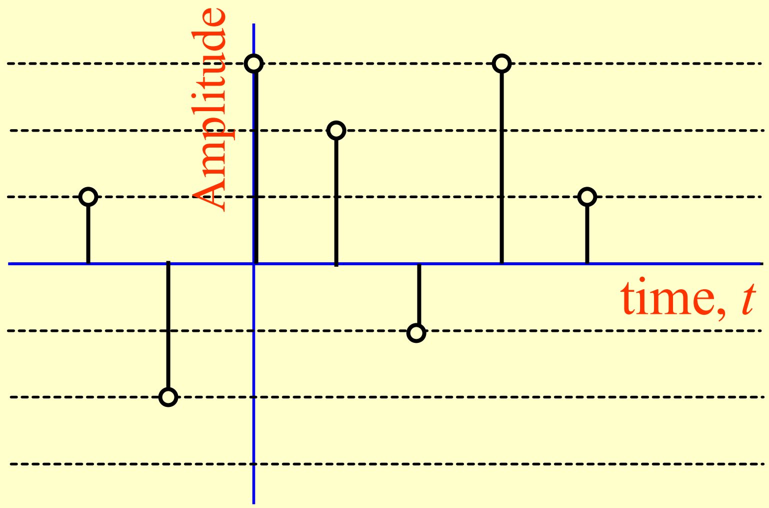
- Signals in a practical digital signal processing system are digital signals obtained by quantizing the sample values either by **rounding** or **truncation**

Discrete-Time Signals: Time-Domain Representation

- Example -



Boxedcar signal



Digital signal

Discrete-Time Signals: Time-Domain Representation

- A discrete-time signal may be a **finite-length** or an **infinite-length sequence**
- Finite-length (also called **finite-duration** or **finite-extent**) sequence is defined only for a finite time interval: $N_1 \le n \le N_2$

  where $-\infty < N_1$ and $N_2 < \infty$ with $N_1 \le N_2$
- **Length** or **duration** of the above finite-length sequence is $N = N_2 - N_1 + 1$

Discrete-Time Signals: Time-Domain Representation

- Example - $x[n] = n^2, -3 \le n \le 4$ is a finite-length sequence of length $4-(-3)+1=8$

$y[n] = \cos 0.4n$ is an infinite-length sequence

Discrete-Time Signals: Time-Domain Representation

- A length-N sequence is often referred to as an N**-point sequence**
- The length of a finite-length sequence can be increased by **zero-padding**, i.e., by **appending** it **with zeros**

Discrete-Time Signals: Time-Domain Representation

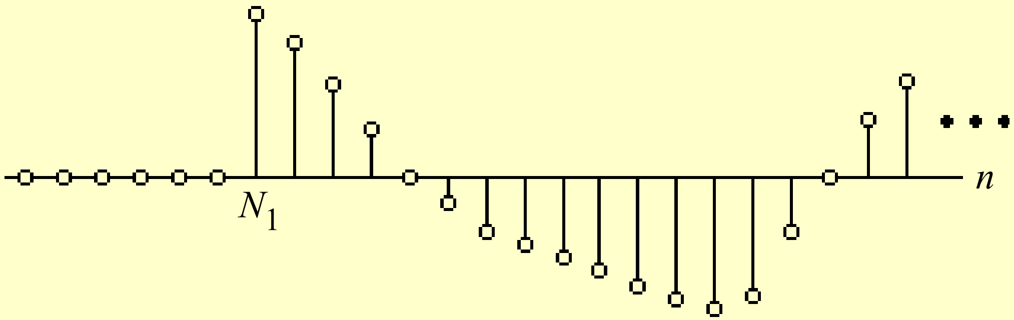
- Example -

$$x_e[n] = \begin{cases} n^2, & -3 \le n \le 4 \\ 0, & 5 \le n \le 8 \end{cases}$$

is a finite-length sequence of length 12 obtained by zero-padding $x[n] = n^2, -3 \le n \le 4$ with 4 zero-valued samples

Discrete-Time Signals: Time-Domain Representation

- A **right-sided sequence** $x[n]$ has zero-valued samples for $n < N_1$

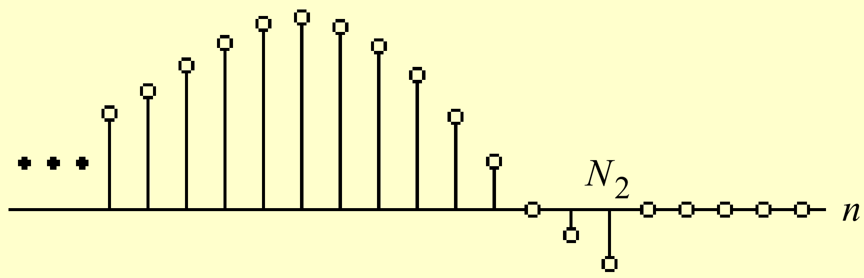


A right-sided sequence

- If $N_1 \geq 0$, a right-sided sequence is called a **causal sequence**

Discrete-Time Signals: Time-Domain Representation

- A **left-sided sequence** $x[n]$ has zero-valued samples for $n > N_2$



A left-sided sequence

- If $N_2 \leq 0$, a left-sided sequence is called a **anti-causal sequence**

Discrete-Time Signals: Time-Domain Representation

"Strength"

- **Size of a Signal**

Given by the norm of the signal

$\mathcal{L}_p$-norm

$$\|x\|_p = \left(\sum_{n=-\infty}^{\infty} |x[n]|^p \right)^{1/p}$$

where p is a positive integer

Discrete-Time Signals: Time-Domain Representation

- The value of p is typically 1 or 2 or ∞

$\mathcal{L}_2$-norm

$$\|x\|_2$$

is the root-mean-squared (rms) value of $\{x[n]\}$

Apart from a sqrt(N) scaling factor: RMS is sqrt(mean(squares)), L2 is sqrt(sum(squares))

Discrete-Time Signals: Time-Domain Representation

$\mathcal{L}_1$-norm $\|x\|_1$

is the mean absolute value of $\{x[n]\}$

Apart from a scaling factor N

$\mathcal{L}_\infty$-norm $\|x\|_\infty$

is the peak absolute value of $\{x[n]\}$, i.e.

$$\|x\|_\infty = |x|_{\max}$$

Discrete-Time Signals: Time-Domain Representation

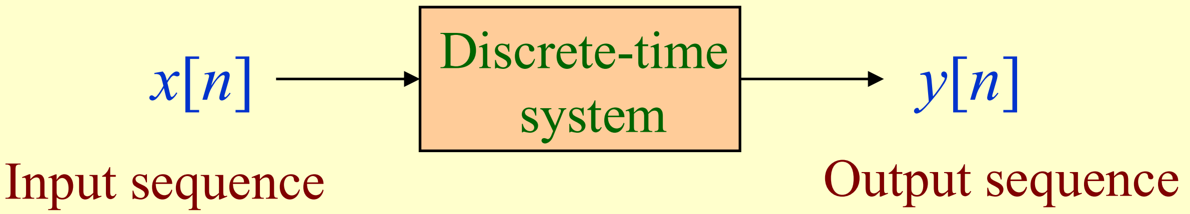
Example

- Let $\{y[n]\}, 0 \leq n \leq N-1$, be an approximation of $\{x[n]\}, 0 \leq n \leq N-1$
- An estimate of the relative error is given by the ratio of the $\mathcal{L}_2$-norm of the difference signal and the $\mathcal{L}_2$-norm of $\{x[n]\}$:

$$E_{rel} = \left(\frac{\sum_{n=0}^{N-1} \left| y[n] - x[n] \right|^2}{\sum_{n=0}^{N-1} \left| x[n] \right|^2} \right)^{1/2}$$

Operations on Sequences

- A single-input, single-output discrete-time system operates on a sequence, called the **input sequence**, according some prescribed rules and develops another sequence, called the output sequence, with more desirable properties

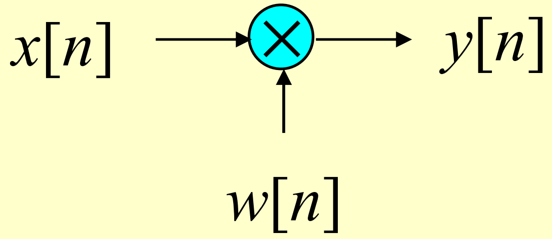


Operations on Sequences

- For example, the input may be a signal corrupted with additive noise
- Discrete-time system is designed to generate an output by removing the noise component from the input
- In most cases, the operation defining a particular discrete-time system is composed of some **elementary operations**

Elementary Operations

- **Product (modulation)** operation:
  - Modulator

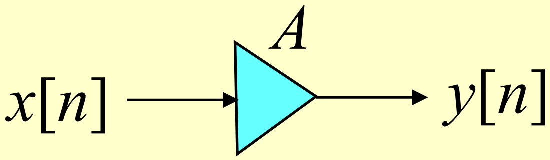


$$y[n] = x[n] \cdot w[n]$$

- An application is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called an **window sequence**
- Process called **windowing**

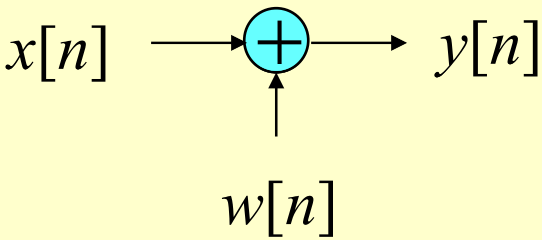
Elementary Operations

- **Multiplication** operation
  - Multiplier



$$y[n] = A \cdot x[n]$$

- Addition operation
  - Adder



$$y[n] = x[n] + w[n]$$

25

Elementary Operations

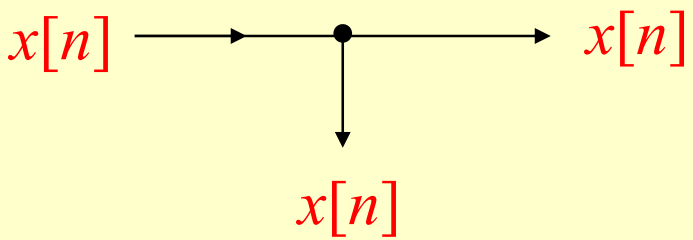
- **Time-shifting** operation: $y[n] = x[n-N]$
  where N is an integer
- If $N > 0$, it is **delaying** operation
  - Unit delay $x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n]$ $\quad y[n] = x[n-1]$
- If $N < 0$, it is an **advance** operation
  - Unit advance $x[n] \longrightarrow \boxed{z} \longrightarrow y[n]$ $\quad y[n] = x[n+1]$

Elementary Operations

- **Time-reversal (folding)** operation:

$$y[n] = x[-n]$$

- **Branching** operation: Used to provide multiple copies of a sequence



Elementary Operations

- Example - Consider the two following sequences of length 5 defined for $0 \leq n \leq 4$:

$$\{a[n]\} = \{3 \quad 4 \quad 6 \quad -9 \quad 0\}$$

$$\{b[n]\} = \{2 \quad -1 \quad 4 \quad 5 \quad -3\}$$

- New sequences generated from the above two sequences by applying the basic operations are as follows:

Elementary Operations

$$\{c[n]\} = \{a[n] \cdot b[n]\} = \{6 \quad -4 \quad 24 \quad -45 \quad 0\}$$

$$\{d[n]\} = \{a[n] + b[n]\} = \{5 \quad 3 \quad 10 \quad -4 \quad -3\}$$

$$\{e[n]\} = \frac{3}{2}\{a[n]\} = \{4.5 \quad 6 \quad 9 \quad -13.5 \quad 0\}$$

- As pointed out by the above example, operations on two or more sequences can be carried out if all sequences involved are of same length and defined for the same range of the time index n

Elementary Operations

"Zero padding"

- However if the sequences are not of same length, in some situations, this problem can be circumvented by appending zero-valued samples to the sequence(s) of smaller lengths to make all sequences have the same range of the time index
- Example - Consider the sequence of length 3 defined for $0 \leq n \leq 2$: $\{f[n]\} = \{-2 \;\; 1 \;\; -3\}$

Elementary Operations

- We cannot add the length-3 sequence $\{f[n]\}$ to the length-5 sequence $\{a[n]\}$ defined earlier
- We therefore first append $\{f[n]\}$ with 2 zero-valued samples resulting in a length-5 sequence $\{f_e[n]\} = \{-2 \quad 1 \quad -3 \quad 0 \quad 0\}$
- Then

$$\{g[n]\} = \{a[n]\} + \{f_e[n]\} = \{1 \quad 5 \quad 3 \quad -9 \quad 0\}$$

Elementary Operations

Ensemble Averaging

- A very simple application of the addition operation in improving the quality of measured data corrupted by an additive random noise
- In some cases, actual uncorrupted data vector **s** remains essentially the same from one measurement to next

Elementary Operations

- While the additive noise vector is random and not reproducible
- Let $\mathbf{d}_i$ denote the noise vector corrupting the i-th measurement of the uncorrupted data vector $\mathbf{s}$:

$$\underbrace{\mathbf{x}_i}_{\text{Measured data vector}} = \underbrace{\mathbf{s}}_{\text{Uncorrupted data vector}} + \underbrace{\mathbf{d}_i}_{\text{Noise vector}}$$

Elementary Operations

- The average data vector, called the ensemble average, obtained after K measurements is given by

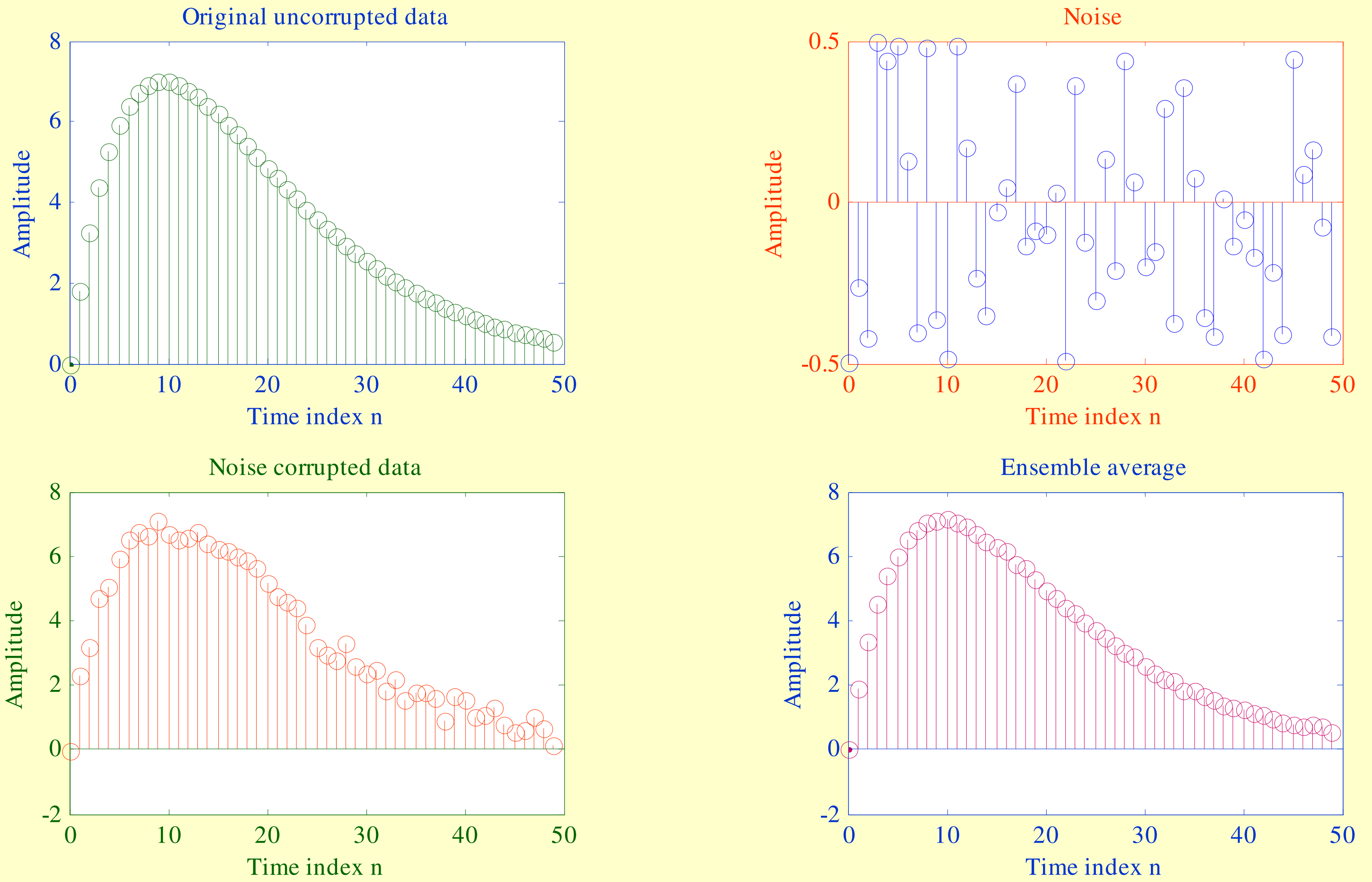
$$\mathbf{x}_{ave} = \frac{1}{K}\sum_{i=1}^{K}\mathbf{x}_i = \frac{1}{K}\sum_{i=1}^{K}(\mathbf{s}+\mathbf{d}_i) = \mathbf{s} + \frac{1}{K}\sum_{i=1}^{K}\mathbf{d}_i$$

- For large values of K, $\mathbf{x}_{ave}$ is usually a reasonable replica of the desired data vector $\mathbf{s}$

Variance of white noise is reduced by K

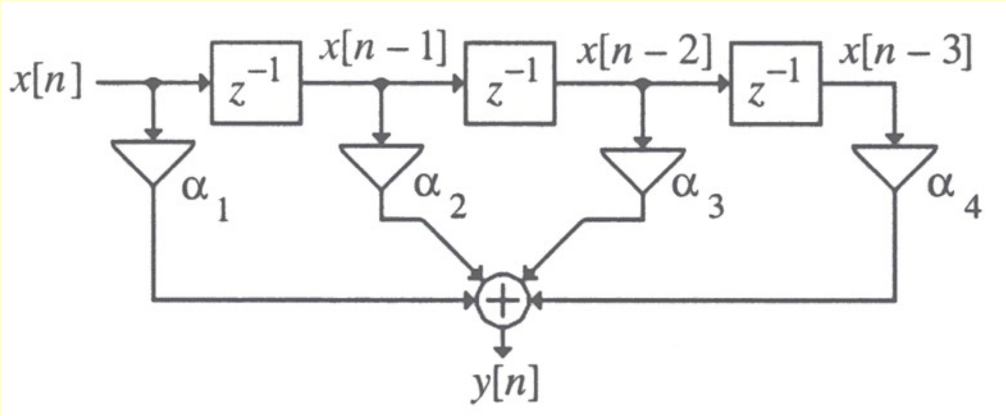
Elementary Operations

- **Example**



Combinations of Basic Operations

- Example -



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

Better: **Sample** Rate Alteration

Sampling Rate Alteration

- Employed to generate a new sequence $y[n]$ with a sampling rate F_T' higher or lower than that of the sampling rate F_T of a given sequence $x[n]$
- **Sampling rate alteration ratio** is $R = \frac{F_T'}{F_T}$

- If $R > 1$, the process called **interpolation**
- If $R < 1$, the process called **decimation**

below we just do **upsampling**

Sampling Rate Alteration

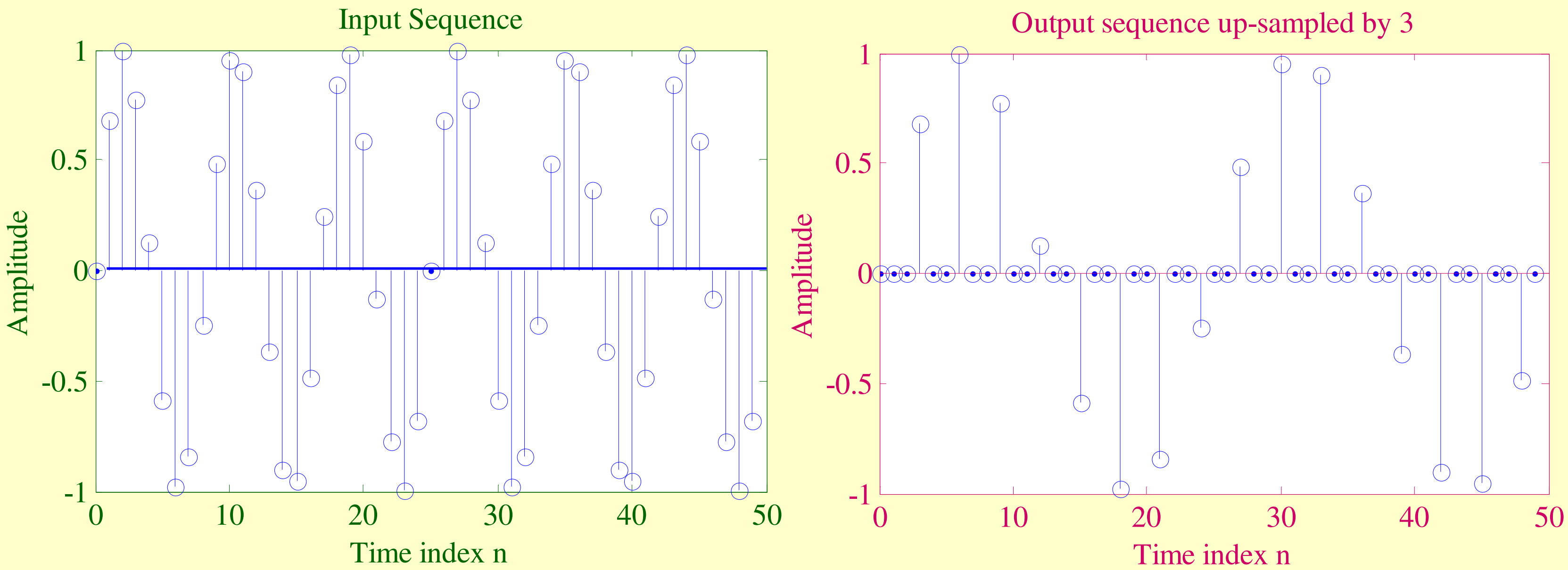
- In **up-sampling** by an integer factor $L > 1$, $L-1$ equidistant zero-valued samples are inserted by the **up-sampler** between each two consecutive samples of the input sequence $x[n]$:

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \cdots \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow x_u[n]$$

Sampling Rate Alteration

- An example of the up-sampling operation



Sampling Rate Alteration

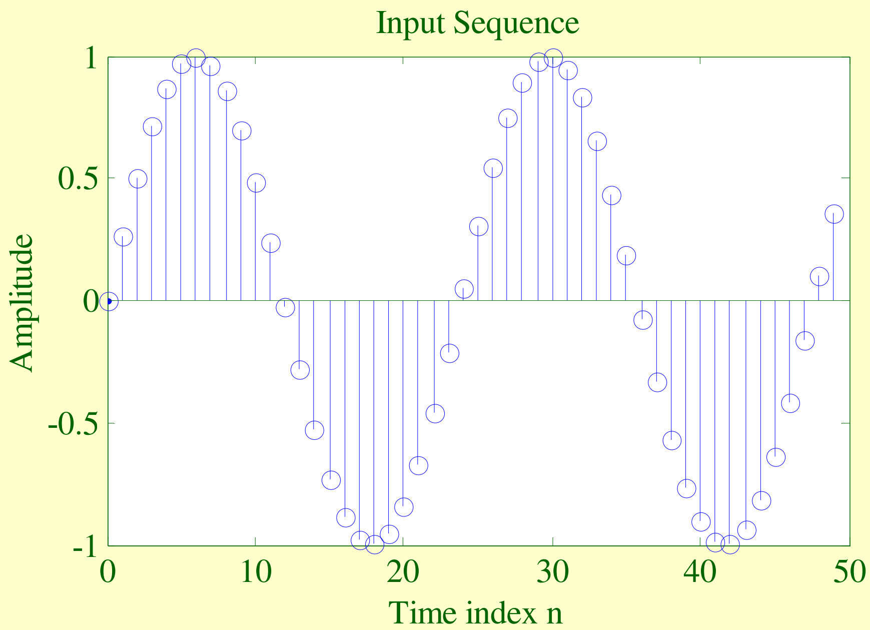
- In **down-sampling** by an integer factor $M > 1$, every M-th samples of the input sequence are kept and $M - 1$ in-between samples are removed:

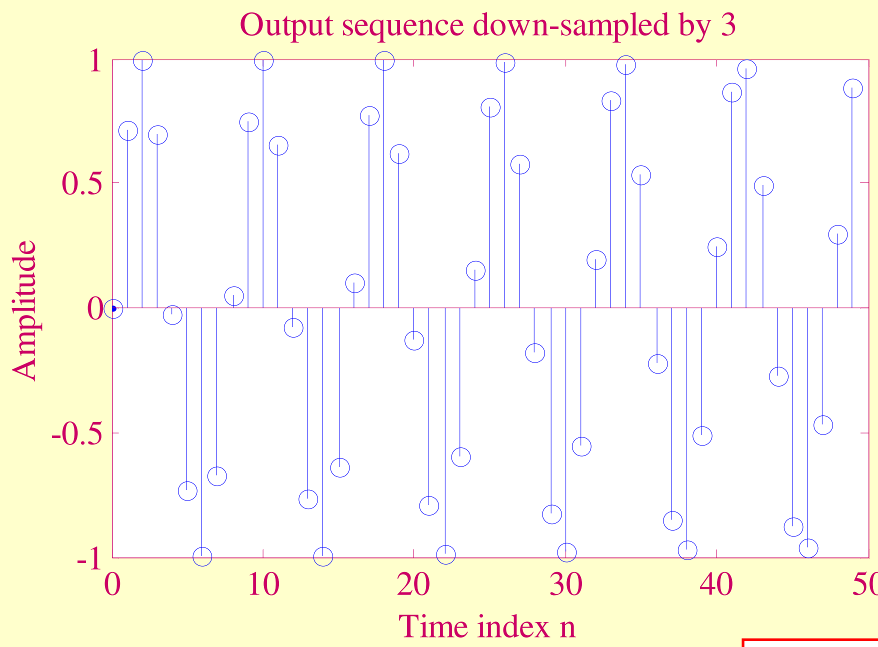
$$y[n] = x[nM]$$

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n]$$

Sampling Rate Alteration

- An example of the down-sampling operation





MATLAB

Classification of Sequences

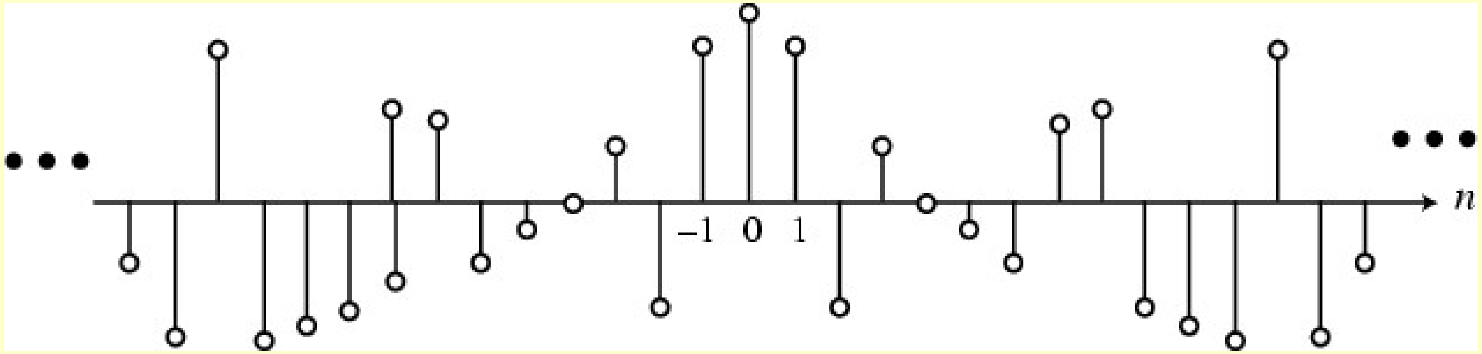
- There are several types of classification
- One classification is in terms of the number of samples defining the sequence
- Another classification is based on its symmetry with respect to time index $n = 0$
- Other classifications in terms of its other properties, such as periodicity, summability, energy and power

Classification of Sequences Based on Symmetry

- **Conjugate-symmetric sequence**:

$$x[n] = x^*[-n]$$

If $x[n]$ is real, then it is an **even sequence**



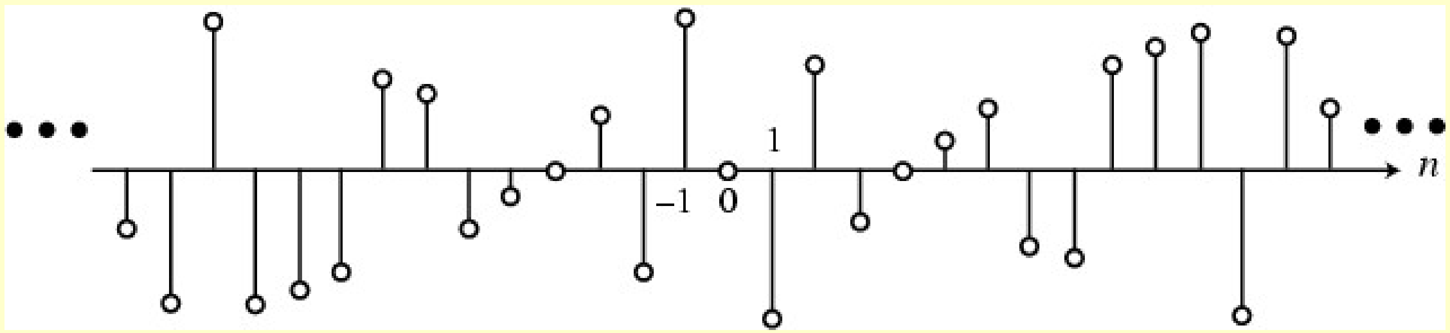
An even sequence

Classification of Sequences Based on Symmetry

- **Conjugate-antisymmetric sequence**:

$$x[n] = -x^*[-n]$$

If $x[n]$ is real, then it is an **odd sequence**



An odd sequence

Classification of Sequences Based on Symmetry

- It follows from the definition that for a conjugate-symmetric sequence $\{x[n]\}$, $x[0]$ must be a real number
- Likewise, it follows from the definition that for a conjugate anti-symmetric sequence $\{y[n]\}$, $y[0]$ must be an imaginary number
- From the above, it also follows that for an odd sequence $\{w[n]\}$, $w[0] = 0$

Classification of Sequences Based on Symmetry

- Any complex sequence can be expressed as a sum of its conjugate-symmetric part and its conjugate-antisymmetric part:

$$x[n] = x_{cs}[n] + x_{ca}[n]$$

where

$$x_{cs}[n] = \tfrac{1}{2}\left(x[n] + x^*[-n]\right)$$

$$x_{ca}[n] = \tfrac{1}{2}\left(x[n] - x^*[-n]\right)$$

Classification of Sequences Based on Symmetry

- Any real sequence can be expressed as a sum of its even part and its odd part:

$$x[n] = x_{ev}[n] + x_{od}[n]$$

where

$$x_{ev}[n] = \frac{1}{2}\left(x[n] + x[-n]\right)$$

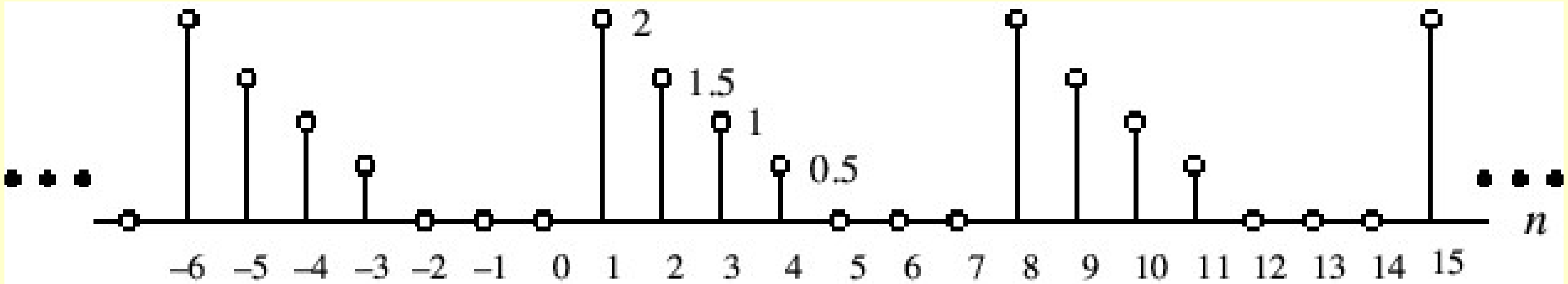
$$x_{od}[n] = \frac{1}{2}\left(x[n] - x[-n]\right)$$

Classification of Sequences Based on Periodicity

- A sequence $\tilde{x}[n]$ satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called a **periodic sequence** with a **period** N where N is a positive integer and k is any integer
- Smallest value of N satisfying $\tilde{x}[n] = \tilde{x}[n + kN]$ is called the **fundamental period**

Classification of Sequences Based on Periodicity

- Example -



- A sequence not satisfying the periodicity condition is called an **aperiodic sequence**

Classification of Sequences Based on Periodicity

- If $\tilde{x}_a[n]$ and $\tilde{x}_b[n]$ are two periodic sequences with fundamental periods N_a and N_b, respectively, then the sequence

$$\tilde{y}[n] = \tilde{x}_a[n] + \tilde{x}_b[n]$$

is a periodic sequence with a fundamental period N given by

$$N = \frac{N_a N_b}{GCD(N_a, N_b)}$$

Greatest Common Divisor

Classification of Sequences: Energy and Power Signals

- Total **energy** of a sequence $x[n]$ is defined by

$$\mathcal{E}_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

- An infinite length sequence with finite sample values may or may not have finite energy
- A finite length sequence with finite sample values has finite energy

Classification of Sequences: Energy and Power Signals

- The **average power** of an aperiodic sequence is defined by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1} \sum_{n=-K}^{K} |x[n]|^2$$

- Define the **energy** of a sequence $x[n]$ over a finite interval $-K \le n \le K$ as

$$\mathcal{E}_{x,K} = \sum_{n=-K}^{K} |x[n]|^2$$

Classification of Sequences: Energy and Power Signals

- Then

$$P_x = \lim_{K \to \infty} \frac{1}{2K+1} \mathcal{E}_{x.K}$$

- The **average power** of a periodic sequence $\tilde{x}[n]$ with a period N is given by

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} \left| \tilde{x}[n] \right|^2$$

- The average power of an infinite-length aperiodic sequence may be finite or infinite

Classification of Sequences: Energy and Power Signals

- Example - Consider the causal sequence defined by

$$x[n] = \begin{cases} 3(-1)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- Note: $x[n]$ has infinite energy
- Its average power is given by

$$P_x = \lim_{K\to\infty} \frac{1}{2K+1}\left(9\sum_{n=0}^{K} 1\right) = \lim_{K\to\infty} \frac{9(K+1)}{2K+1} = 4.5$$

Classification of Sequences: Energy and Power Signals

- An infinite energy signal with finite average power is called a **power signal**

  Example - A periodic sequence which has a finite average power but infinite energy

- A finite energy signal with zero average power is called an **energy signal**

These definitions are relevant only for **infinite length** sequences

Other Types of Classifications

- A sequence $x[n]$ is said to be **bounded** if

$$|x[n]| \leq B_x < \infty$$

- Example - The sequence $x[n] = \cos 0.3\pi n$ is a bounded sequence as

$$|x[n]| = |\cos 0.3\pi n| \leq 1$$

Other Types of Classifications

- A sequence $x[n]$ is said to be **absolutely summable** if
$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$
- Example - The sequence
$$y[n] = \begin{cases} 0.3^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$
is an absolutely summable sequence as
$$\sum_{n=0}^{\infty} |0.3^n| = \frac{1}{1-0.3} = 1.42857 < \infty$$

Other Types of Classifications

- A sequence $x[n]$ is said to be **square-summable** if

$$\sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 < \infty$$

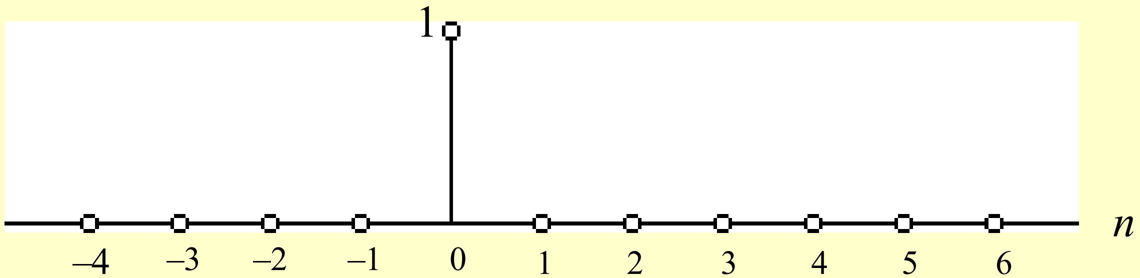
- Example - The sequence

$$h[n] = \frac{\sin 0.4n}{\pi n}$$

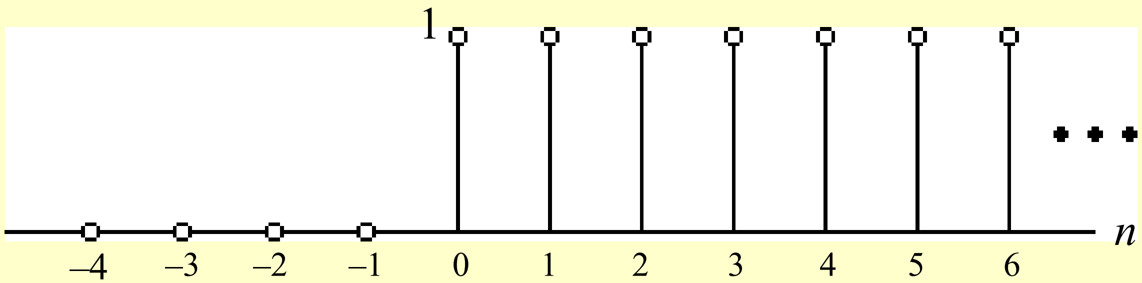
is square-summable but not absolutely summable

Basic Sequences

- **Unit sample sequence** - $\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$



- **Unit step sequence** - $\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$



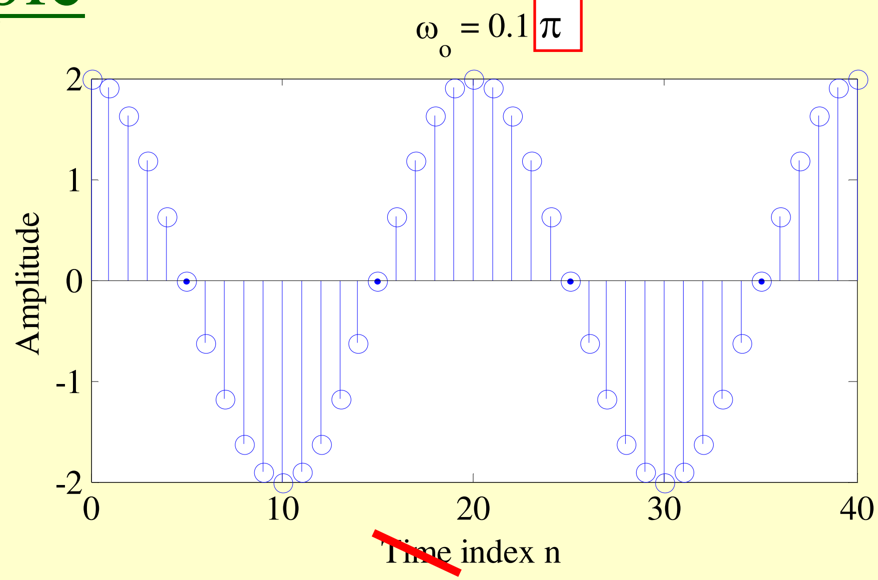
Basic Sequences

- **Real sinusoidal sequence** -

$$x[n] = A\cos(\omega_o n + \phi)$$

where A is the **amplitude**, ω_o is the **angular frequency**, and ϕ is the **phase** of $x[n]$

Example -



Basic Sequences

- **Exponential sequence -**

$$x[n] = A\alpha^n, \; -\infty < n < \infty$$

where A and α are real or complex numbers

- If we write $\alpha = e^{(\sigma_o + j\omega_o)}, A = |A| e^{j\phi}$,

then we can express

$$x[n] = |A| e^{j\phi} e^{(\sigma_o + j\omega_o)n} = x_{re}[n] + j\, x_{im}[n],$$

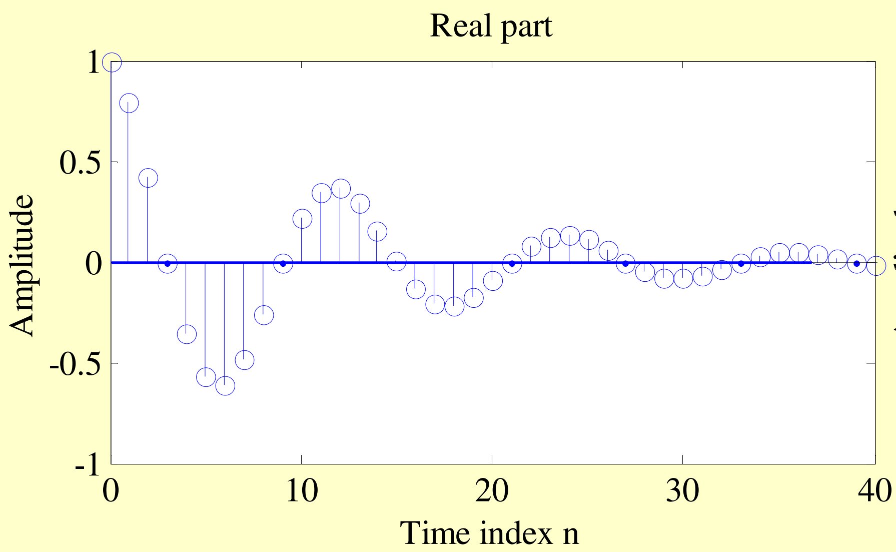
where

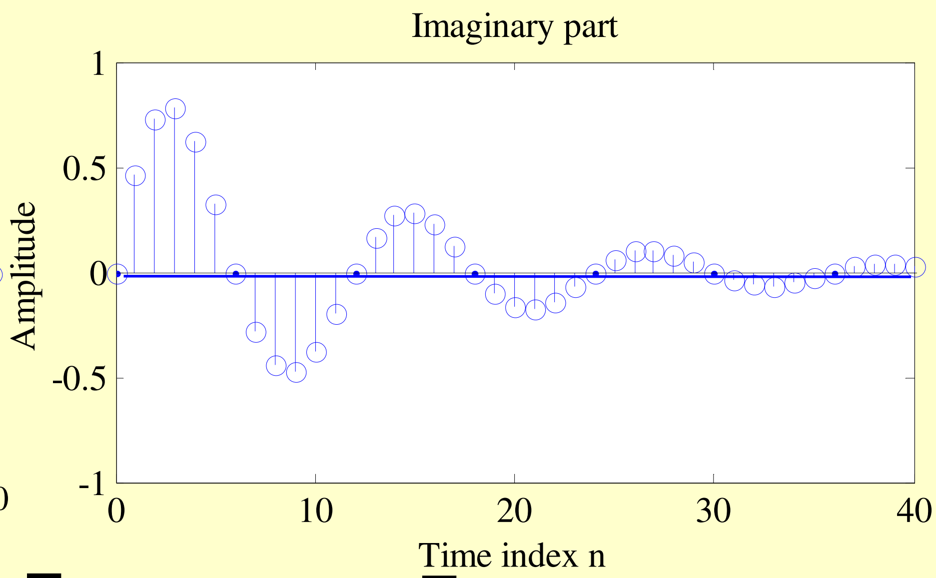
$$x_{re}[n] = |A| e^{\sigma_o n} \cos(\omega_o n + \phi),$$

$$x_{im}[n] = |A| e^{\sigma_o n} \sin(\omega_o n + \phi)$$

Basic Sequences

- $x_{re}[n]$ and $x_{im}[n]$ of a complex exponential sequence are real sinusoidal sequences with constant $(\sigma_o = 0)$, growing $(\sigma_o > 0)$, and decaying $(\sigma_o < 0)$ amplitudes for $n > 0$





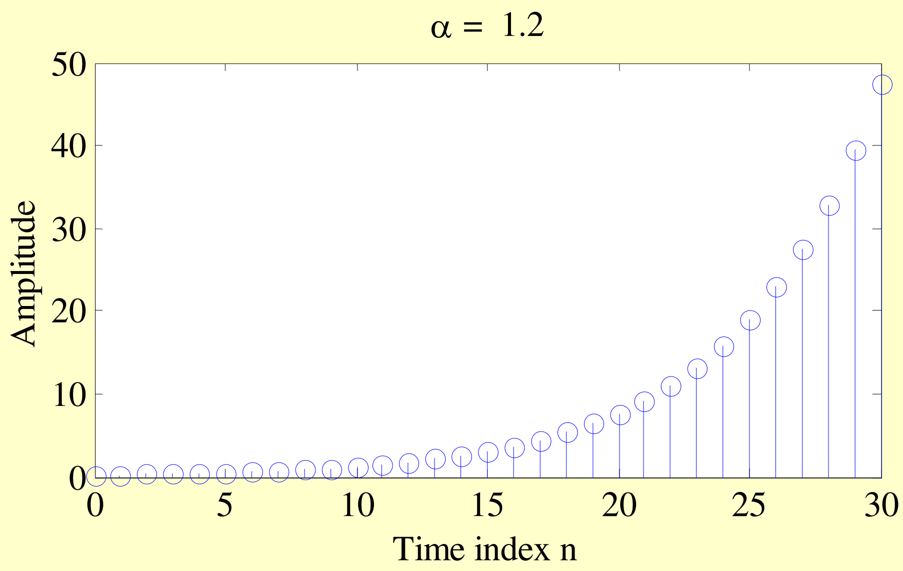
$$x[n] = \exp\left[\left(-\frac{1}{12} + j\frac{\pi}{6}\right)n\right]$$

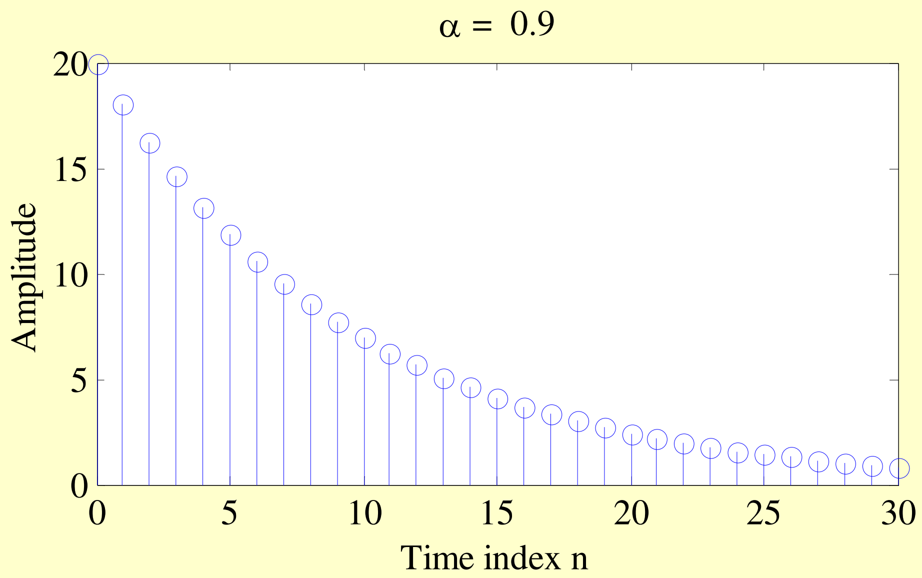
Basic Sequences

- **Real exponential sequence -**

$$x[n] = A\alpha^n, \quad -\infty < n < \infty$$

where A and α are real numbers





Basic Sequences

- Sinusoidal sequence $A\cos(\omega_o n + \phi)$ and complex exponential sequence $B\exp(j\omega_o n)$ are periodic sequences of period N if $\omega_o N = 2\pi r$ where N and r are positive integers
- Smallest value of N satisfying $\omega_o N = 2\pi r$ is the **fundamental period** of the sequence
- To verify the above fact, consider

$$x_1[n] = \cos(\omega_o n + \phi)$$
$$x_2[n] = \cos(\omega_o (n + N) + \phi)$$

Basic Sequences

- Now $x_2[n] = \cos(\omega_o(n+N)+\phi)$

$$= \cos(\omega_o n + \phi)\cos\omega_o N - \sin(\omega_o n + \phi)\sin\omega_o N$$

which will be equal to $\cos(\omega_o n + \phi) = x_1[n]$ only if

$$\sin\omega_o N = 0 \quad \text{and} \quad \cos\omega_o N = 1$$

- These two conditions are met if and only if

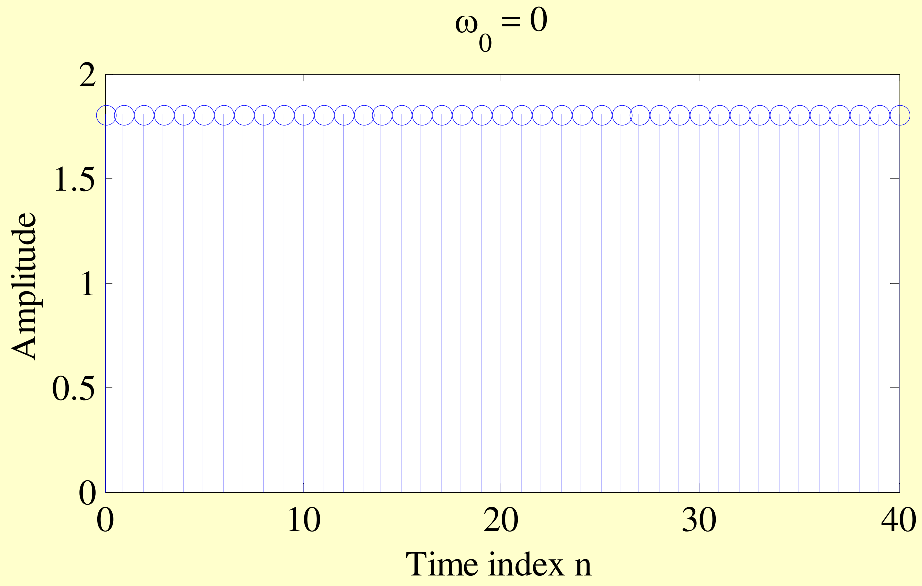
$$\omega_o N = 2\pi r \quad \text{or} \quad \frac{2\pi}{\omega_o} = \frac{N}{r}$$

Basic Sequences

We might say that:
N is the true period
N/r is the apparent period (the one of the implied continuous function)

- If $2\pi/\omega_o$ is a noninteger rational number, then the period will be a multiple of $2\pi/\omega_o$ (i.e., r>1)
- Otherwise, the sequence is **aperiodic**
- Example - $x[n] = \sin(\sqrt{3}n + \phi)$ is an aperiodic sequence

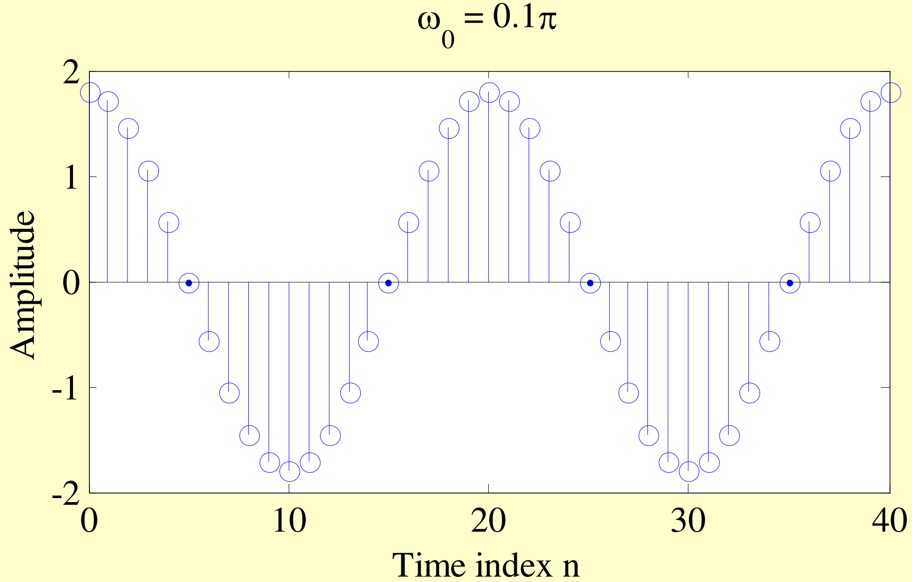
Basic Sequences



- Here $\omega_o = 0$

- Hence period $N = \frac{2\pi r}{0} = 1$ for $r = 0$

Basic Sequences



- Here $\omega_o = 0.1\pi$

- Hence $N = \dfrac{2\pi r}{0.1\pi} = 20$ for $r = 1$

Basic Sequences

- Property 1 - Consider $x[n] = \exp(j\omega_1 n)$ and $y[n] = \exp(j\omega_2 n)$ with $0 \le \omega_1 < \pi$ and $2\pi k \le \omega_2 < \boxed{\pi(2k+1)}$ where k is any positive integer
- If $\omega_2 = \omega_1 + 2\pi k$, then $x[n] = y[n]$
- Thus, $x[n]$ and $y[n]$ are indistinguishable

Basic Sequences

- Property 2 - The frequency of oscillation of $A\cos(\omega_o n)$ increases as ω_o increases from 0 to π, and then decreases as ω_o increases from π to 2π
- Thus, frequencies in the neighborhood of $\omega = 0$ are called **low frequencies**, whereas, frequencies in the neighborhood of $\omega = \pi$ are called **high frequencies**

Basic Sequences

- Because of Property 1, a frequency ω_o in the neighborhood of $\omega = 2\pi\, k$ is indistinguishable from a frequency $\omega_o - 2\pi k$ in the neighborhood of $\omega = 0$

  and a frequency ω_o in the neighborhood of $\omega = \pi(2k+1)$ is indistinguishable from a frequency $\omega_o - \pi(2k+1)$ in the neighborhood of $\omega = \pi$

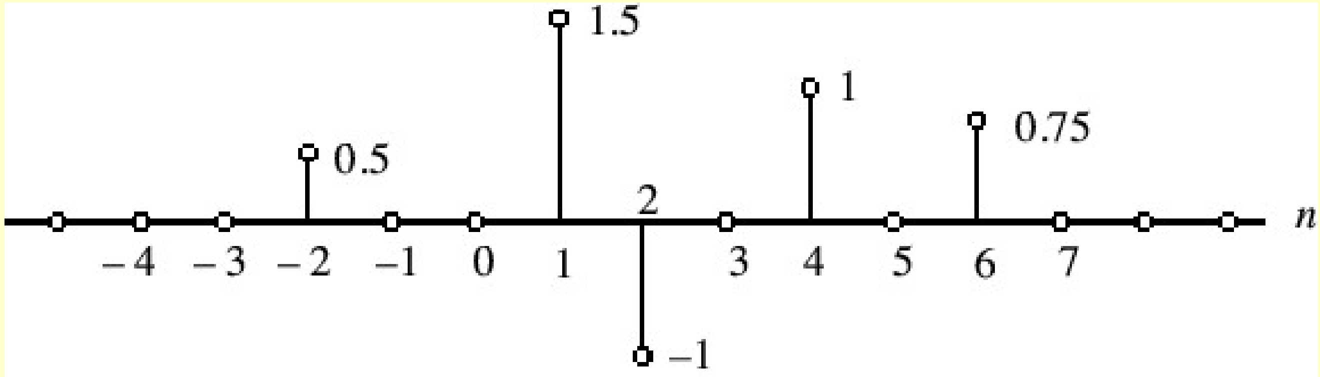
Basic Sequences

- Frequencies in the neighborhood of $\omega = 2\pi\, k$ are usually called **low frequencies**
- Frequencies in the neighborhood of $\omega = \pi\,(2k+1)$ are usually called **high frequencies**
- $v_1[n] = \cos(0.1\pi n) = \cos(1.9\pi n)$ is a low-frequency signal
- $v_2[n] = \cos(0.8\pi n) = \cos(1.2\pi n)$ is a high-frequency signal

MATLAB

Basic Sequences

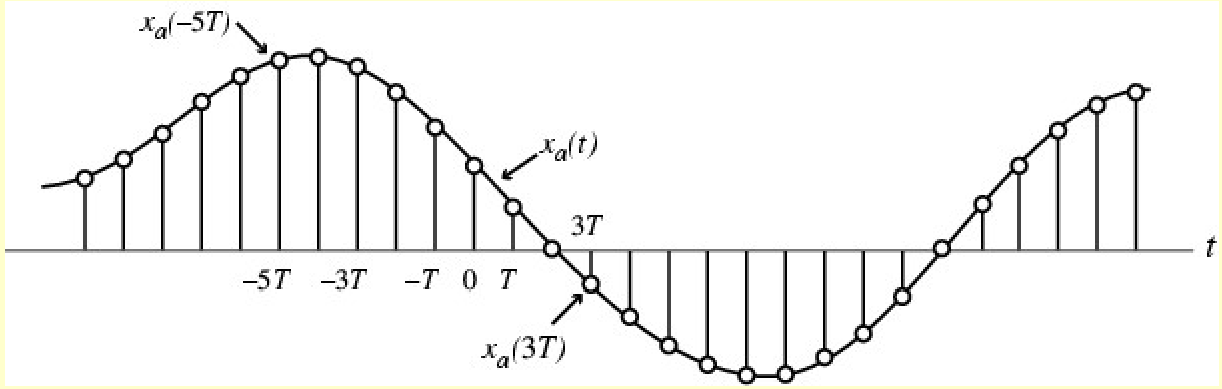
- An arbitrary sequence can be represented in the time-domain as a weighted sum of some basic sequence and its delayed (advanced) versions



$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2]$$
$$+ \delta[n-4] + 0.75\delta[n-6]$$

The Sampling Process

- Often, a discrete-time sequence $x[n]$ is developed by uniformly sampling a continuous-time signal $x_a(t)$ as indicated below



- The relation between the two signals is

$$x[n] = x_a(t)\big|_{t=nT} = x_a(nT),\ n = \ldots, -2, -1, 0, 1, 2, \ldots$$

The Sampling Process

- Time variable t of $x_a(t)$ is related to the time variable n of $x[n]$ only at discrete-time instants t_n given by

$$t_n = nT = \frac{n}{F_T} = \frac{2\pi n}{\Omega_T}$$

with $F_T = 1/T$ denoting the sampling frequency and

$\Omega_T = 2\pi F_T$ denoting the sampling angular frequency

The Sampling Process

- Consider the continuous-time signal

$$x_a(t) = A\cos(2\pi f_o t + \phi) = A\cos(\Omega_o t + \phi)$$

- The corresponding discrete-time signal is

$$x[n] = A\cos(\Omega_o nT + \phi) = A\cos(\frac{2\pi\Omega_o}{\Omega_T} n + \phi)$$

$$= A\cos(\omega_o n + \phi)$$

where $\boxed{\omega_o = 2\pi\Omega_o / \Omega_T = \Omega_o T}$

is the normalized digital angular frequency of $x[n]$

Note that according to the value of T:

- an analog sine function generates different sampled sequences
- different sine functions can generate the same sampled sequence

The Sampling Process

- If the unit of sampling period T is in seconds
- The unit of normalized digital angular frequency ω_o is radians/sample
- The unit of [illegible] analog angular frequency Ω_o is radians/second
- The unit of analog frequency f_o is hertz (Hz)

The Sampling Process

- The three continuous-time signals

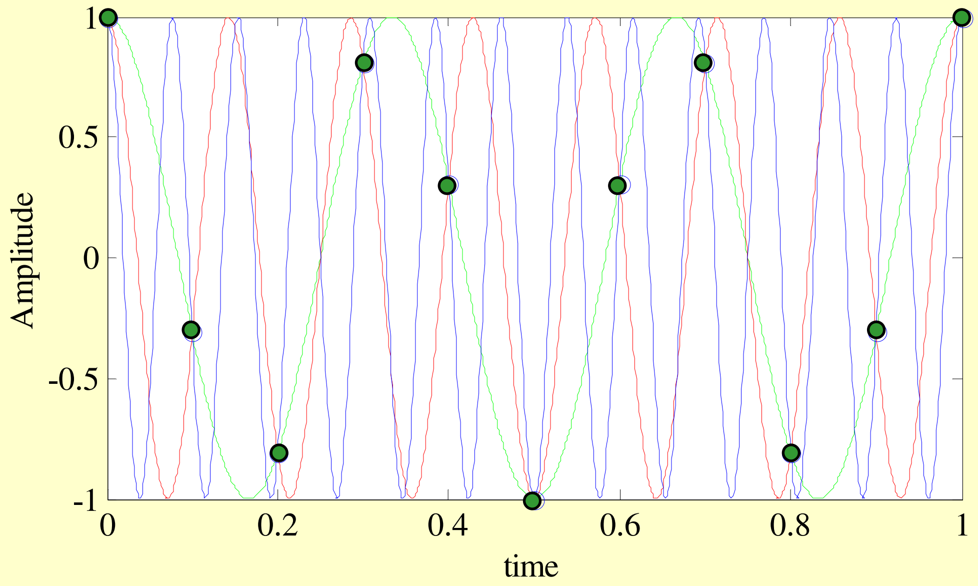
$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

of frequencies 3 Hz, 7 Hz, and 13 Hz, are sampled at a sampling rate of 10 Hz, i.e. with $T = 0.1$ sec. generating the three sequences

$$g_1[n] = \cos(0.6\pi n) \qquad g_2[n] = \cos(1.4\pi n)$$
$$g_3[n] = \cos(2.6\pi n)$$

The Sampling Process

- Plots of these sequences (shown with circles) and their parent time functions are shown below:



- Note that each sequence has exactly the same sample value for any given *n*

The Sampling Process

- This fact can also be verified by observing that

$$g_2[n] = \cos(1.4\pi n) = \cos\big((2\pi - 0.6\pi)n\big) = \cos(0.6\pi n)$$

$$g_3[n] = \cos(2.6\pi n) = \cos\big((2\pi + 0.6\pi)n\big) = \cos(0.6\pi n)$$

- As a result, all three sequences are identical and it is difficult to associate a unique continuous-time function with each of these sequences

The Sampling Process

- The above phenomenon of a continuous-time signal of higher frequency acquiring the identity of a sinusoidal sequence of lower frequency after sampling is called **aliasing**

The Sampling Process

- Since there are an infinite number of continuous-time signals that can lead to the same sequence when sampled periodically, additional conditions need to imposed so that the sequence $\{x[n]\} = \{x_a(nT)\}$ can uniquely represent the parent continuous-time signal $x_a(t)$
- In this case, $x_a(t)$ can be fully recovered from $\{x[n]\}$

The Sampling Process

- Example - Determine the discrete-time signal $v[n]$ obtained by uniformly sampling at a sampling rate of 200 Hz the continuous-time signal

$$v_a(t) = 6\cos(60\pi t) + 3\sin(300\pi t) + 2\cos(340\pi t) + 4\cos(500\pi t) + 10\sin(660\pi t)$$

- Note: $v_a(t)$ is composed of 5 sinusoidal signals of frequencies 30 Hz, 150 Hz, 170 Hz, 250 Hz and 330 Hz

The Sampling Process

- The sampling period is $T = \frac{1}{200} = 0.005$ sec
- The generated discrete-time signal $v[n]$ is thus given by

$$v[n] = 6\cos(0.3\pi n) + 3\sin(1.5\pi n) + 2\cos(1.7\pi n)$$
$$+ 4\cos(2.5\pi n) + 10\sin(3.3\pi n)$$
$$= 6\cos(0.3\pi n) + 3\sin\big((2\pi - 0.5\pi)n\big) + 2\cos\big((2\pi - 0.3\pi)n\big)$$
$$+ 4\cos\big((2\pi + 0.5\pi)n\big) + 10\sin\big((4\pi - 0.7\pi)n\big)$$

The Sampling Process

$$= 6\cos(0.3\pi n) - 3\sin(0.5\pi n) + 2\cos(0.3\pi n) + 4\cos(0.5\pi n) - 10\sin(0.7\pi n)$$

$$= 8\cos(0.3\pi n) + 5\cos(0.5\pi n + 0.6435) - 10\sin(0.7\pi n)$$

- Note: $v[n]$ is composed of 3 discrete-time sinusoidal signals of normalized angular frequencies: 0.3π, 0.5π, and 0.7π

The Sampling Process

- Note: An identical discrete-time signal is also generated by uniformly sampling at a 200-Hz sampling rate the following continuous-time signals:

$$w_a(t) = 8\cos(60\pi t) + 5\cos(100\pi t + 0.6435) - 10\sin(140\pi t)$$

$$g_a(t) = 2\cos(60\pi t) + 4\cos(100\pi t) + 10\sin(260\pi t)$$
$$+ 6\cos(460\pi t) + 3\sin(700\pi t)$$

The Sampling Process

- Recall $\omega_o = \dfrac{2\pi\Omega_o}{\Omega_T}$

- Thus if $\Omega_T > 2\Omega_o$, then the corresponding normalized digital angular frequency ω_o of the discrete-time signal obtained by sampling the parent continuous-time sinusoidal signal will be in the range $-\pi < \omega < \pi$

- No aliasing

The Sampling Process

- On the other hand, if $\Omega_T < 2\Omega_o$, the normalized digital angular frequency will foldover into a lower digital frequency $\omega_o = \langle 2\pi\Omega_o / \Omega_T \rangle_{2\pi}$ in the range $-\pi < \omega < \pi$ because of aliasing
- Hence, to prevent aliasing, the sampling frequency Ω_T should be greater than 2 times the frequency Ω_o of the sinusoidal signal being sampled

The Sampling Process

- Generalization: Consider an arbitrary continuous-time signal $x_a(t)$ composed of a weighted sum of a number of sinusoidal signals
- $x_a(t)$ can be represented uniquely by its sampled version $\{x[n]\}$ if the sampling frequency Ω_T is chosen to be greater than 2 times the highest frequency contained in $x_a(t)$

The Sampling Process

- The condition to be satisfied by the sampling frequency to prevent aliasing is called the **sampling theorem**
- A formal proof of this theorem will NOT be presented later