

Exact Histogram Specification

Dinu Coltuc, Philippe Bolon, and Jean-Marc Chassery

Abstract—While in the continuous case, statistical models of histogram equalization/specification would yield exact results, their discrete counterparts fail. This is due to the fact that the cumulative distribution functions one deals with are not exactly invertible. Otherwise stated, exact histogram specification for discrete images is an ill-posed problem. Invertible cumulative distribution functions are obtained by translating the problem in a K -dimensional space and further inducing a strict ordering among image pixels. The proposed ordering refines the natural one. Experimental results and statistical models of the induced ordering are presented and several applications are discussed: image enhancement, normalization, watermarking, etc.

Index Terms—Exact histogram equalization, exact histogram specification, strict ordering.

I. INTRODUCTION

HISTOGRAM specification (or modeling) refers to a class of image transforms which aims to obtain images the histograms of which have a desired shape [1]–[3]. Even if specifying a meaningful histogram for a certain image is not obvious, there are some general ones (such as uniform, Gaussian, exponential) whose usefulness is clearly understood. Thus, obtaining a uniform histogram image corresponds to the well-known image enhancement technique called histogram equalization. By means of histogram equalization, graylevels are spread over the entire scale and an equal number of pixels is allocated to each graylevel. For human observers, this yields more balanced and better contrasted images. Furthermore, equalized images, besides their pleasant appearance, make details visible in dark or bright regions of the original images. Better results in image enhancement are obtained if the human visual system (HVS) is taken into account. The image histogram is specified according to a certain model of the HVS such that the subjectively perceived image has an equalized histogram. Several models of the HVS have been taken into account [4], [5].

Besides image enhancement, histogram specification is of interest in many other image processing tasks. For example, most thresholding/segmentation algorithms are based on mixtures of Gaussian probability density functions and optimal schemes are expected to be obtained if such conditions are met. Similarly, optimal coding could be obtained if exact histogram specification

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D. Coltuc is with the Department of Electrical Engineering, Valahia University of Targoviste, 0200 Targoviste, Dambovită, Romania (e-mail: coltuc@valahia.ro).

Ph. Bolon is with the LISTIC, University of Savoie, 74016 Annecy cedex, France (e-mail: philippe.bolon@esia.univ-savoie.fr).

J.-M. Chassery is with the LIS, National Polytechnic Institute of Grenoble, 38402 Saint-Martin d'Heres cedex, France (e-mail: jean-marc.chassery@inpg.fr).

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TABLE I
PROBABILITY OF PIXEL EQUALITY (GAUSSIAN-LIKE DISTRIBUTION)

$P(k)$	upper bound	lower bound
$P(2)$	$5.62\sigma^{-2}10^{-2}$	$3.97\sigma^{-2}10^{-2}$
$P(3)$	$9.16\sigma^{-3}10^{-3}$	$5.61\sigma^{-3}10^{-3}$
$P(4)$	$1.29\sigma^{-4}10^{-3}$	$7.91\sigma^{-4}10^{-4}$
$P(5)$	$1.82\sigma^{-5}10^{-4}$	$7.89\sigma^{-5}10^{-5}$
$P(6)$	$2.57\sigma^{-6}10^{-5}$	$1.11\sigma^{-6}10^{-5}$

were available. Finally, exact histogram specification immediately yields image normalization.

Histogram specification can be directly approached as an optimization problem: Given the original image histogram and the desired one, find a graylevel mapping to obtain the best approximation of the desired histogram. Such a mapping can be found by simply grouping graylevels in order to minimize the approximation error to the desired histogram [6]. Other solutions have been investigated as well, for instance by using the graph theory [7]. Although direct approaches are intuitive and straightforward, statistical modeling not only gives the mapping but also a sound understanding of the histogram specification problem.

According to the classical approach to image enhancement by histogram specification, image intensity is regarded as a continuous random variable (RV) characterized by its probability density function (PDF). In this setting, given a RV with a known distribution, the function (transform) to be found must be such that the transformed RV has the specified PDF. In the sequel, the approach of [1] is briefly recalled.

For example, in the case of histogram equalization, let \mathbf{r} be a continuous RV supposed to take values in $[0, 1]$ and let $p_r(r)$ be its PDF. If $T(r) = \int_{-\infty}^r p_r(v) dv$, i.e., the cumulative distribution function (CDF) of \mathbf{r} , is strictly increasing, the RV $\mathbf{z} = T(\mathbf{r})$ is uniformly distributed in $[0, 1]$ (see [8]). Given a continuous graylevel image f taking values in $[0, L]$, the normalized image f/L takes values in $[0, 1]$. If the normalized image is considered as the RV \mathbf{r} , the RV \mathbf{z} obtained as above is uniformly distributed in $[0, 1]$, $L \times \mathbf{z}$ is uniform in $[0, L]$ and, thus, the transform $r \rightarrow L \times T(r)$ equalizes the graylevel image.

Histogram specification generalizes the histogram equalization case. As before, the continuous setting is considered. Let \mathbf{r} be the original RV and let \mathbf{u} be a RV having the desired PDF, $p_u(u)$. Let T and G be the CDFs of \mathbf{r} and \mathbf{u} , respectively. Both T and G are supposed to be strictly increasing. Let furthermore $\mathbf{z} = T(\mathbf{r})$ and $\mathbf{v} = G(\mathbf{u})$. Since \mathbf{z} and \mathbf{v} are uniform in $[0, 1]$, one can impose $\mathbf{v} = \mathbf{z}$ and therefore, $\mathbf{u} = G^{-1}(\mathbf{z}) = G^{-1}(T(\mathbf{r}))$. Thus, $G^{-1}(T)$ is proven to be the desired function which maps the given \mathbf{r} into the desired \mathbf{u} to recover PDF $p_u(u)$. Even if G^{-1} cannot usually be given by a closed formula, the problem can be solved numerically.

While in the continuous case the specification/equalization algorithms are supposed to provide exact results, their discrete

TABLE II
STRICT ORDERING PROBABILITY (GAUSSIAN-LIKE DISTRIBUTION)

σ	N	B	$P(2)$	$P(3)$	$P(4)$	$P(5)$	$P(6)$
5	256	<i>ub</i>	0.000000	0.230971	0.959504	0.999176	0.999977
5	256	<i>lb</i>	0.000000	0.091344	0.934722	0.998097	0.999946
10	256	<i>ub</i>	0.000002	0.832619	0.997420	0.999974	1.000000
10	256	<i>lb</i>	0.000000	0.741462	0.995790	0.999940	0.999999
15	256	<i>ub</i>	0.003112	0.947171	0.999490	0.999997	1.000000
15	256	<i>lb</i>	0.000285	0.915183	0.999167	0.999992	1.000000
20	256	<i>ub</i>	0.038894	0.977363	0.999839	0.999999	1.000000
20	256	<i>lb</i>	0.010133	0.963299	0.999736	0.999998	1.000000
30	256	<i>ub</i>	0.236209	0.993239	0.999968	1.000000	1.000000
30	256	<i>lb</i>	0.129927	0.988982	0.999948	1.000000	1.000000
5	512	<i>ub</i>	0.000000	0.002813	0.847317	0.996701	0.999907
5	512	<i>lb</i>	0.000000	0.000068	0.762956	0.992397	0.999785
10	512	<i>ub</i>	0.000000	0.479911	0.989698	0.999897	0.999999
10	512	<i>lb</i>	0.000000	0.301534	0.983232	0.999762	0.999997
15	512	<i>ub</i>	0.000000	0.804506	0.997957	0.999986	1.000000
15	512	<i>lb</i>	0.000000	0.701019	0.996665	0.999969	1.000000
20	512	<i>ub</i>	0.000002	0.912316	0.999353	0.999997	1.000000
20	512	<i>lb</i>	0.000000	0.860830	0.998944	0.999993	1.000000
30	512	<i>ub</i>	0.003078	0.973176	0.999872	1.000000	1.000000
30	512	<i>lb</i>	0.000280	0.956569	0.999791	0.999999	1.000000

all

counterparts fail. Thus, the discrete equalization algorithm cannot separate among equal graylevel pixels in order to get a perfectly flat histogram. As it is well-known, the discrete histogram equalization becomes a point-wise transform on the graylevel scale. Whatever the original image histogram, the resulted histogram is flattened, but may be far from being uniform. Some graylevels are merged together to approximate the bins of a uniform histogram. The resulted graylevels are spread as uniformly as possible covering the whole range up to the white level. Obviously, the discrete specification algorithm gives approximate results, too. While for histogram equalization the mapping immediately appears as the CDF of the original image distribution, the mapping derivation problem is more difficult for the general case of histogram specification [1], [2], [10].

Several attempts have been made so far to improve histogram equalization/specification performances [2], [6], [9], [11]. For instance, the conventional algorithm is further refined to get exact histogram equalization by randomly separating pixels [2], [9]. Exact uniform histograms are achieved at the expense of noisy images as stated in [2]. A better solution was proposed in [11], where the histogram approximation is improved avoiding noise by separating pixels according to their local mean on the four horizontal and vertical neighbors. We shall further refine this latter solution.

In this paper, an approach to exact histogram specification for real images is proposed. It extends our previous work on strict ordering on discrete images [13] and introduces a theoretical analysis framework. The paper is organized as follows. The basic principle of exact histogram specification is presented in Section II. In Section III, the ordering relation is defined. The theoretical analysis and experimental results concerning ordering are provided in Section IV. In Section V, applications to image processing are investigated. Finally, conclusions are given in Section VI.

II. EXACT HISTOGRAM SPECIFICATION

The discrete version of the statistical approaches could have yielded exact results (perfectly equalized/specified histograms)

if CDFs had been invertible [1]. In the discrete case, CDFs are staircase functions, hence they are not invertible except in the case when pixels take distinct values. Since the number of pixels in an image is usually considerably larger than the number of graylevels, the distinct pixel value case is irrelevant. The CDF of an RV z determines the probabilities $P\{z \leq z\}$ and consequently, it depends on the ordering relation used. Otherwise stated, a discrete exact histogram equalization/specification problem is solved if the usual ordering is replaced by a new ordering relation which induces a strict ordering among image pixels. (Notice that a strict ordering reduces the problem to the distinct pixel case.)

A. Principle

Let f be a discrete $N \times M$ image having L graylevels and let $H = \{h_0, h_1, \dots, h_{L-1}\}$ be the histogram to be specified. Notice that H is the nonnormalized image histogram, i.e., h_l is the number of pixels having graylevel l . Let us further suppose that an ordering relation, \prec , is defined among the pixels of f such that the induced ordering is strict. Then, the exact histogram specification simply proceeds as follows [13].

- 1) Order image pixels:

$$f(x_1, y_1) \prec f(x_2, y_2) \prec \dots \prec f(x_{NM}, y_{NM}). \quad (1)$$

- 2) Split the ordered string (1) from left to right in L groups, such as group j has h_j pixels.
- 3) For all the pixels in a group j , assign graylevel j .

The exact equalization algorithm considers groups of NM/L pixels in step 2).

The aforementioned scheme yields exact results, namely the image is transformed to obtain exactly the desired histogram, provided that such a histogram is a valid one. The validity of histograms is understood as the equality between the image size (number of pixels) and the sum of histogram bins, i.e.,

$$\sum_{i=0}^{L-1} h(i) = N \times M. \quad (2)$$

Specifying a histogram is equivalent to specifying a certain distribution whose PDF is exactly the normalized image histogram. Since histogram bins take integer values, for an $N \times M$ size image, PDFs cannot be specified at a resolution better than

$$\epsilon = \frac{1}{MN}. \quad (3)$$

In other words, given any desired continuous distribution, a $N \times M$ image can be transformed to approximate it with the precision ϵ defined in (3).

Equation (1) requires strict inequalities. On the other hand, the histogram specification algorithm described previously does not require an absolutely strict ordered sequence; it simply requires to discriminate among L groups of pixels. Otherwise stated, problems appear when equal graylevel pixels have to be separated (have to be assigned to different graylevels). Besides, even if two pixels or a small group of equal pixels have to be split into two distinct groups, the error is not significant. Therefore, we can relax the condition of strict ordering to almost strict ordering. In fact, we could generally accept that some small groups of pixels are equal in the sense of the considered ordering.

III. ORDERING

The discrete exact histogram specification is solved if a strict ordering can be induced on image pixels. Such a strict ordering can be obtained in many ways. For instance, any one-to-one mapping between image coordinates and a set of integers, $O : [1, N] \times [1, M] \rightarrow [1, MN]$, induces a strict ordering. Thus, $f(x_1, y_1)$ can be considered to be greater than $f(x_2, y_2)$ in the sense of the ordering induced by the mapping O , if $O(x_1, y_1) > O(x_2, y_2)$ with the usual ordering on the set of integers. The number of such mappings is of the order of $(NM)!$, but most of them are useless for exact histogram specification. In order to have a useful strict ordering, the induced ordering must be consistent with the normal ordering; i.e., if a pixel graylevel is greater than another one with the normal ordering on integers, it should be greater with the new ordering as well. Otherwise stated, the new ordering should refine the normal ordering on the set of integers. Regarding refinement, the induced ordering should correspond, in a certain way, to the human perception of brightness; otherwise results would become noisy (see [2]). All constraints stated above are intended to preserve the image content, as it is perceived by humans.

In order to induce such an ordering the pixel neighborhood is taken into account. In fact, this idea has already been used to improve histogram equalization by taking into account the graylevel average on the four neighbors in horizontal and vertical directions [11]. Since only the average on the four neighbors does not succeed to completely discriminate among equal pixels, we elaborate on this idea by considering a family of neighborhoods around each pixel.

Let K be a fixed integer and let $W_i, i = 1, \dots, K$, be a family of closed neighborhoods such that

$$W_1 \subset W_2 \subset \dots \subset W_K. \quad (4)$$

For each pixel $f(x, y)$, let $m_i(x, y)$ be the mean value of the graylevels of f on W_i . We assume that the image is expanded by replicating the border pixels in order to accommodate W_K on its borders. Let $M(x, y)$ denote the K -tuple $(m_1(x, y), m_2(x, y), \dots, m_K(x, y))$ and let us further consider the lexicographic order defined on these K -tuples. Let us recall that $M(a, b)$ is less than $M(c, d)$ according to the lexicographic order if $m_1(a, b) < m_1(c, d)$ or if there is $j, 1 \leq j \leq K$ such that $m_i(a, b) = m_i(c, d)$ for $i = 1, \dots, j-1$ and $m_j(a, b) < m_j(c, d)$.

The order defined by the lexicographic ordering induces a complete ordering on the set $M(x, y)$ of K -tuples. Since a K -tuple has been associated to each pixel, there is a correspondence between the set of K -tuples and the image. Therefore, the same ordering can be extended to discrete images as well. We will further write $f(x_1, y_1) \prec f(x_2, y_2)$ when $M(x_1, y_1)$ is less than $M(x_2, y_2)$ with respect to the lexicographic order.

To summarize, by using a vector operator, the problem is transferred from a scalar image to a K -dimensional space by associating a vector to each pixel and then, by lexicographically ordering the vectors the same order is induced among the image pixels. The vector operator can be seen as a filter bank

$$\Phi = (\phi_1, \phi_2, \dots, \phi_K) \quad (5)$$

and

$$\Phi(f)(x, y) = (\phi_1(f), \phi_2(f), \dots, \phi_K(f))(x, y) \quad (6)$$

Each ϕ_i extracts some local information about graylevels around the current pixel (x, y) . The inclusion among the filter supports of (4) is intended to somehow order the amount of information extracted by each filter. Thus, when i is small, the information extracted is strongly connected to the current pixel. As index i increases, support W_i increases as well and the weight of the current pixel decreases in the filter response. This is a reason for ranking pixels using the lexicographic order starting with the first index.

We consider the $\phi_i, i = 1, \dots, K$, as moving average filters. Other linear or nonlinear filters can be used as well (Gaussian filters, median, etc).

In order to completely specify the Φ operator, filter supports must be specified. If W_1 is of size 1×1 and $K = 1$, the usual ordering on $[0, L]$ is recovered. Let $K = 2$; in this case, the order induced on the image is refined. Thus, one can find a strict ordering among pixels having the same graylevel (or alternatively, the same mean value over W_1), but different mean values over W_2 . The greater K , the finer the ordering. Therefore, an ordering consistent with the usual one is obtained if we restrict W_1 to 1×1 (which implies that ϕ_1 is the identity filter).

Under such weak constraints, there are many possibilities to select the W_i family, and thus, the Φ operator. The objective in our selection was to keep filter supports as small as possible. This choice reduces computational complexity. Moreover it prevents pixels far apart from the current pixel from influencing its rank in the ordered string. Meanwhile, the W_i family should have some geometrical meaning and symmetry. Therefore a family of moving average filters is designed starting with ϕ_1 having a one-pixel size support and then, enlarging from the next ones while keeping the symmetry for a minimum increase

of the filter support. What follows is the description of the first six filter masks:

$$\begin{aligned}\phi_1 &= [1] & \phi_2 &= \frac{1}{5} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \phi_3 &= \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \phi_4 &= \frac{1}{13} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \\ \phi_5 &= \frac{1}{21} \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \\ \phi_6 &= \frac{1}{25} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.\end{aligned}$$

The development of the family can continue, on the same basis, with ϕ_7, ϕ_8 , and so on, keeping symmetry and the minimum increase between filter supports.

IV. ORDERING EVALUATION

With the proposed ordering relation, a pixel turns out to be brighter than another pixel when its local mean is greater than the local mean of the other one. The initial ordering of the graylevels is refined. Our aim is to achieve a strict ordering, or, in a less restrictive setting, a strict ordering almost everywhere, i.e., having very few equalities in (1).

Obviously, the induced ordering depends on K as well as on the image: Original graylevel distribution, graylevel range and image size. For images with very large uniform areas (like synthetic images), a strict ordering may not be possible. We will assume we deal with natural images having enough graylevels and enough details (or noise). A too large value for K means an increase in the computational complexity of the ordering procedure. Moreover, when K is increased, the rank of a pixel depends on pixels located far apart (which is of no physical relevance). Therefore, a moderate value for K is desired.

A. Theoretical Analysis

In order to quantify the rather fuzzy measures given above, namely moderate size and enough gray levels, the simplified model of images having quantized Gaussian IID (independent identically distributed) pixels is considered. The probability of equal pixels as a function of K and σ is evaluated. Notice that the variance of the Gaussian distribution is closely related to the number of graylevels of the image: since the probability of having values outside the 3σ range situated around the mean of the Gaussian is almost zero, the graylevel range L can be considered to be about 6σ .

With the proposed ordering, there is equality between two pixels of coordinates (x_1, y_1) and (x_2, y_2) , respectively, if

$$\phi_i(f(x_1, y_1)) = \phi_i(f(x_2, y_2)) \quad (7)$$

for $i = 1, \dots, K$. Since ϕ_i are moving average filters, equalities between averages stand for equalities of graylevel sums over the corresponding neighborhoods. Thus

$$\sum_{(x,y) \in W_i(x_1, y_1)} f(x, y) = \sum_{(x,y) \in W_i(x_2, y_2)} f(x, y) \quad (8)$$

where $i = 1, \dots, K$. Due to the neighborhood inclusion, once there is sum equality for W_i , it is not necessary to verify the equality for W_{i+1} , but only for the set difference $W_{i+1} - W_i$. The previous observation leads to the replacement of each W_i by a set V_i , where $V_1 = W_1$ and

$$V_{i+1} = W_{i+1} - W_i \quad (9)$$

for $i > 1$. By replacing the W_i family and discarding the normalization in moving average filtering, the operator $\Phi = (\phi_1, \dots, \phi_K)$ is replaced by the equivalent operator $\Psi = (\psi_1, \dots, \psi_K)$, where

$$\begin{aligned}\psi_1 &= [1] & \psi_2 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} & \psi_3 &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ \psi_4 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} & \psi_5 &= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \\ \psi_6 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.\end{aligned}$$

The equality of two pixels from (7) becomes

$$\psi_i(f(x_1, y_1)) = \psi_i(f(x_2, y_2)), \quad i = 1, \dots, K \quad (10)$$

which is equivalent to

$$\sum_{(x,y) \in V_i(x_1, y_1)} f(x, y) = \sum_{(x,y) \in V_i(x_2, y_2)} f(x, y), \quad i = 1, \dots, K. \quad (11)$$

In order to compute the probability of having pixel equality according to the proposed ordering relation, one has to determine the probability of equality between sums of pixels. Let P_m be the probability of equality between the sum of m pixels. Original image pixel distribution is denoted by $p_1(x)$. Let $p_m(x)$ be the probability law of the sum of m RV. For $m = 1$, the probability P_1 that pixels $f(x_1, y_1)$ and $f(x_2, y_2)$ have the same graylevel is

$$P_1 = \sum_{i=0}^{L-1} p_1(i)p_1(i) = \sum_{i=0}^{L-1} p_1^2(i). \quad (12)$$

In the general case, the probability to have equality between two sums of m independent random variables is

$$P_m = \sum_{i=0}^{mL-m} p_m(i)^2 \quad (13)$$

where the probability law $p_m(x)$ is computed by convolution

$$p_m = p_1 * p_1 * \dots * p_1 = p_1 * p_{m-1} \quad (14)$$

Obviously, $P_1 \geq P_2 \geq P_3 \geq \dots \geq P_m$.

Let P_{V_i} be the probability to have equality for the corresponding sums of pixels on V_i neighborhoods and $P(K)$ be the probability of equality of two pixels as a function of K . $P(K)$ immediately follows as a mere product:

$$P(K) = \prod_{i=1}^K P_{V_i}. \quad (15)$$

It should be noted that the probability of equality between 2 pixels, $P(K)$, depends not only on K , but also on the pixel locations. This is due to the fact that if the two pixels under consideration are close together, some common pixels can appear in the computation of P_{V_i} probabilities. Thus, the probability of sum equalities increases. Therefore, for a given K , $P(K)$ varies between a lower and an upper bound. In the sequel, let $K \leq 6$.

The lower bound is obtained if no pixels are common in the computation of P_{V_i} . In this case, $P_{V_1} = P_1, P_{V_2} = P_{V_3} = P_{V_4} = P_{V_6} = P_4$, and $P_{V_5} = P_8$.

The upper bound is obtained if pixels are adjacent along diagonal directions. Thus, For P_{V_1} one has P_1 , as above. Next, when V_2 is centered on the two diagonal pixels, one has 2 pixels in common. Since two pixels are common, their sum can be eliminated from the sum of 4 pixels on each V_2 . Thus P_{V_2} increases from the probability to have equality between sums of 4 RV's up to the probability to have equality between sums of 2 RV's, i.e., $P_{V_2} = P_2$. Furthermore, the V_3 neighborhood of each pixel contains the other current pixel, therefore only three pixels do count, yielding $P_{V_3} = P_3$. No differences appear for P_{V_4} and P_{V_6} , i.e., one has $P_{V_4} = P_{V_6} = P_4$.

For P_{V_5} , two out of eight pixels are common, and there is equality between the sums of the two pixels which are also contained in V_2 neighborhoods. Therefore, $P_{V_5} = P_4$. (When line or column adjacency is considered, one has $P_{V_1} = P_1, P_{V_2} = P_{V_3} = P_3, P_{V_4} = P_{V_6} = P_4$ and $P_{V_5} = P_8$, which yields a slightly lower probability.)

The case of a Gaussian-like graylevel distribution is further considered

$$p_1(x) = \frac{\lambda}{\sqrt{2\pi}\sigma} \exp^{-\frac{(x-m)^2}{2\sigma^2}} \quad (16)$$

where $x = 0, 1, \dots, L-1$ and $\lambda \approx 1$. Probabilities $p_k, P_k, P(k)$ can be directly computed by using (14)(15)–(16). Alternatively, they can be approximated by considering the continuous case. In the sequel, in order to get closed formulas, the latter approach is considered.

For two independent Gaussian random variables, $N(m_a, \sigma_a)$ and $N(m_b, \sigma_b)$, the distribution of the sum is Gaussian as well, $N(m, \sigma)$, with: $m = m_a + m_b$ and $\sigma^2 = \sigma_a^2 + \sigma_b^2$. Then, if

$p_1 = N(m, \sigma)$, it immediately follows that $p_i = N(m_i, \sigma_i)$, where $m_i = im, \sigma_i = \sqrt{i}\sigma$, and $1 < i \leq 8$.

The P_i probabilities are

$$P_i = \sum_x \left(\frac{1}{\sqrt{2\pi}\sigma_i} \exp^{-\frac{(x-m_i)^2}{2\sigma_i^2}} \right)^2 = \frac{1}{2\sqrt{\pi}\sigma_i} \sum_x \frac{1}{\sqrt{2\pi}(\sigma_i/\sqrt{2})} \exp^{-\frac{(x-m_i)^2}{2(\sigma_i/\sqrt{2})^2}}. \quad (17)$$

Since the final sum corresponds to a distribution $N(m_i, \sigma_i/\sqrt{2})$, one has

$$\sum_x \frac{1}{\sqrt{2\pi}(\sigma_i/\sqrt{2})} \exp^{-\frac{(x-m_i)^2}{2(\sigma_i/\sqrt{2})^2}} \approx 1. \quad (18)$$

It follows:

$$P_i = \frac{1}{2\sqrt{\pi}\sigma_i}. \quad (19)$$

Furthermore, by using (15) and the relation between P_{V_i} and P_j discussed above, the lower and upper bounds of $P(K)$ are computed. It can be seen that the upper bound is approximately twice the lower bound and, as K increases, both probabilities become very small. The results for $K = 2, \dots, 6$ are presented in Table I.

For an $N \times M$ image, there are $R = \binom{2}{NM}$ pixel pairs. If the probability to have two equal pixels according to the proposed ordering is $P(K)$, the probability to have two distinct pixels is $1 - P(K)$. Then, the probability to have all pairs distinct is

$$P \approx (1 - P(K))^R. \quad (20)$$

Using (20) and Table I, the lower and upper bounds of the probability of strict ordering can be evaluated. They depend on

- 1) the number of neighborhoods K ;
- 2) the image size $N \times M$;
- 3) the graylevel range L , or equivalently, σ .

Thus, the probability increases with K and decreases with $N \times M$ and $L(\sigma)$. In Table II, the results for two typical image sizes (256×256 and 512×512), $\sigma \in [5, 30]$ and $2 \leq K \leq 6$ are presented. In each case, the probability is evaluated by considering the bounds (B), lower (lb) and upper (ub), of $P(K)$. The results are given with a seven-decimal digit precision. The error is less than 5×10^{-7} which is considerably smaller than ϵ , namely 1.5×10^{-5} for $N = M = 256$ and 3.8×10^{-6} for $N = M = 512$.

Several conclusions can be drawn from Table II. The most important result is that a reasonably small value of K , i.e., $K = 6$, assures strict ordering. Next, the differences between the results obtained with the lower and upper bounds of $P(K)$ are not significant as soon as $K > 4$. The results obtained for $\sigma = 5$ are not of great concern—the image is supposed to have less than 30 graylevels range and consequently, a poor graylevel resolution. As soon as σ increases, the Gaussian distribution model gives very good results.

The probabilities of sum equality in the case of a uniform distribution model are considerably lower than those obtained



Fig. 1. (Left) Original and (right) perfectly equalized (right) test images.

for the Gaussian one and obviously, the strict ordering probability is higher. Thus, for an image having L graylevels, one has $p_1(x) = (1)/(L)$ and, hence, $P_1(x) = (1)/(L)$. Next, $p_2 = p_1 * p_1$ has a triangular shape. P_2 can be directly computed; (using $\sum_{i=1}^n i^2 = (n-1)n(2n+1)/6$); it follows that $P_2 = (2L^2+1)/(3L^3) \approx (2)/(3L)$. For $i \geq 3$, p_i can be very well approximated by a Gaussian distribution; the error decreases with i (central limit theorem). Thus, $p_3 = p_1 * p_2 = N(3L/2, L/2)$ and $P_3 \approx (1)/(\sqrt{\pi}L)$, $p_4 = p_2 * p_2 = N(2L, L/\sqrt{3})$ and $P_4 \approx (\sqrt{3})/(2\sqrt{\pi}L)$, finally $p_8 = p_4 * p_4 = N(4L, L\sqrt{2}/\sqrt{3})$, $P_8 \approx (\sqrt{3})/(2\sqrt{2\pi}L)$.

B. Experimental Results

The statistical analysis shows that strict ordering is achieved for $K \geq 6$. Ordering evaluation on **real images** gives very good results, too. Almost strict ordering is induced for **$K = 6$** . For instance, with the image “Lena” of size **256×256** and with the new ordering for $K = 5$, there are only 8 pairs of equal pixels and for $K = 6$, the ordering is strict. As expected, on the same image, but of size **512×512** , there is a small decrease in performance. There are 352 pairs of equal pixels for $K = 5$ and the number of equalities decreases to **six pairs for $K = 6$** . Quite similar results have been found in all the tests performed so far.



Fig. 2. (Left) Logarithmic and (right) linear histogram specified test images.

In the worst case, for 512×512 size images, a couple of tens of pairs of nonseparable pixels have been found. Compared with the image size (262 144 pixels), this means that almost completely strict ordering is achieved.

In real images, the statistical independence of pixels is generally not satisfied. Conversely, **pixels are correlated and this increases the probability of equality**. However, in the light of the results obtained so far, the ordering obtained for $K = 6$ is appropriate for any application. A number of at most tens of equal pixel pairs compared with the image size of 262 144 pixels, means a very good separation of image pixels and has no practical influence on the specification result if pixel pairs differ from the interval limits in the ordered string. Thus, the burden of increasing K does not make any sense.

V. APPLICATIONS

The immediate use of exact histogram equalization/specification is to replace its classical counterpart in some applications where improvements are expected as, for instance, exact image normalization or image enhancement. New specific applications are foreseen, for example, image watermarking or histogram equalization inversion.

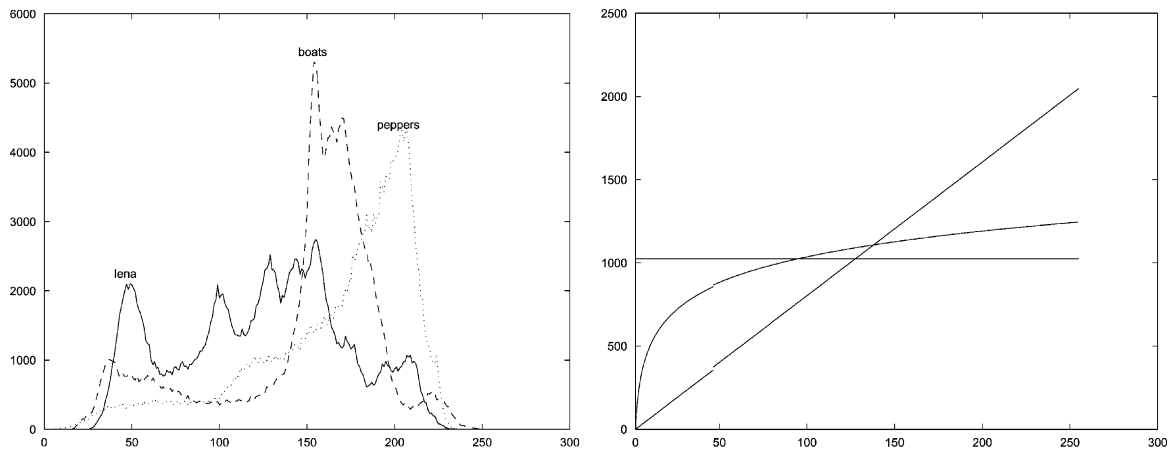


Fig. 3. True histograms before and after exact specification.

Fig. 4. Local histogram equalization: (left) 16×16 window; (right) 32×32 window.

A. Image Enhancement

Histogram equalization/specification is mainly used for image enhancement. For instance, in Fig. 1, the exact histogram equalization of test images is presented. The same test images having linear and logarithmic histogram are presented in Fig. 2. Compared with the exact equalization case, the transformed images turn out to be biased to white levels. (We stress that images shown in Fig. 1 (right column) and Fig. 2 have exactly uniform, linear and logarithmic histograms as shown in Fig. 3). Regarding image enhancement, it should be noticed that **exact histogram specification allows the precise implementation of complex human visual histogram modeling techniques** (see, for instance, [4]).

Since image statistics may change drastically from one region to another, local approaches have proven to give better results than global ones. Contrast enhancement by **local (adaptive) histogram specification** has received much attention in the literature [14]–[16]. **Local exact histogram specification is straightforward: A sliding window is considered and, for each window location, the ordering and histogram specification are performed, but only the value of the central pixel is kept.** Two examples of local histogram equalization for windows of size

16×16 (left) and 32×32 (right), respectively, are shown in Fig. 4. Several comments should be made. First, local histogram equalization considerably increases the contrast—this is why such methods are used in medical imaging. Second, local minima and maxima are firmly forced into black and white, respectively. This is the reason why many details are enhanced (for instance, white or black lines in the boat image). For almost constant regions, the contrast increase generates noise; the smaller the window, the bigger the noise. A final remark concerning the image of Fig. 4 (left) advocates somehow the importance of rank in image processing. Since the window size is 16×16 , the number of pixels in the window (256 pixels) is equal to the number of graylevels. Therefore, in Fig. 4(a), the graylevel of each pixel is exactly its local rank in the window. It can be seen that the image information content is well preserved by pixel local rank.

Extending histogram specification to color images is not straightforward. Following the proposed approach, one should define a strict ordering relation among color image pixels. An immediate solution is to transfer the processing of color images to simply graylevel ones by representing images in a color space where one coordinate is intensity (luminance) and then



Fig. 5. Color images exact histogram specification: (top) original, (middle) logarithmic, and (bottom) linear.

to **process only the luminance component**. Such color spaces are so-called television color spaces (YUV , YIQ , YC_aC_b), perceptual color spaces (HSI, CIELAB), etc., [17]. Let **HSI** (hue, saturation, intensity) be such a color space. Since images are generally represented in RGB color space, exact histogram specification is addressed by: i) conversion from RGB to the HSI, ii) ordering, iii) exact histogram specification performed on the I component (like for graylevel images) and finally, and

iv) HSI to RGB conversion. By ordering on the I component, the hypothesis of natural order refinement discussed above holds. Besides, by histogram specification on the I component, no color shift occurs. An example of exact histogram specification for color images is shown in Fig. 5.

The proposed scheme is consistent with some classical methods which perform histogram specification on luminance in order to avoid color shifts (see, for instance, [18]). Our ap-

proach allows fine histogram tuning thanks to exact histogram specification. We mention that histogram equalization/specification directly in RGB color space has been approached as well [19].

B. Other Specific Applications

1) Image Normalization: Exact histogram specification provides a procedure for real image normalization. By specifying a uniform histogram one obtains images normalized with respect to i) histogram (uniform histogram), ii) graylevel average ($L/2$), iii) energy, and iv) entropy (8 bits/pixel). **Other distributions could be of interest for image normalization such as, for instance, Gaussian or mixture of Gaussians, Laplacian, etc.**

2) Histogram Specification Inversion: In the framework of classical histogram specification or equalization, the recovery of the original image is an unsolved problem. With the proposed approach, this problem turns out to be exact histogram specification of the original histogram for the transformed image. The solution is exactly the original image under the hypothesis that ordering among pixels is preserved by exact histogram specification. Since the hypothesis of order preservation does not completely hold, we expect the reconstruction not to be identical with the original. Obviously, the histogram of the recovered image is exactly the original histogram.

The restored image is a very good approximation of the original. As an example, we have considered original recovering after classical histogram equalization. Thus, for image “Lena” we have found less than 4% erroneous pixels (i.e., 10 343 out of 262 144); and a PSNR of 58.5 dB [20].

3) Invisible Watermarking: Another application of exact histogram specification is image watermarking in the spatial domain: the signature is inserted in the histogram (or it is the histogram itself), marking becomes exact histogram specification and the detection basically consists of histogram computation [21], [22]. The choice of the signature determines whether the watermarking is fragile or robust. By specifying histograms for which compact graylevel intervals are eliminated or considerably reduced, robust watermarking (resistant to JPEG compression, linear and nonlinear filtering and notably robust against geometrical distortions) is obtained.

VI. CONCLUSION

An ill-posed problem, exact histogram specification, is solved. Our approach is based on the definition of an ordering relation which induces almost strict ordering on image pixels. Theoretical and experimental results on the existence of the strict ordering are provided. Once ordering is achieved, pixels are immediately separated into classes and assigned to the desired graylevel. The proposed strict ordering is consistent with the natural one and thus, the information content of images is generally preserved.

An immediate application of the proposed technique is to replace classical histogram equalization and specification. The proposed approach allows direct verification of, for instance, image enhancement by human visual models histogram specification. Exact histogram specification allows very precise image normalization, which is of general interest in image processing.

Recently, the use of exact histogram specification for image watermarking was investigated with very promising results.

The proposed ordering principle is general. It is not restricted to a specific filter bank or graylevel range. For instance, Gaussian filters or combinations of Gaussians and Laplacians could yield an ordering which better matches the human visual system. The use of a bank of gradient or Laplacian filters provides the ordering of potential contour pixels and hence, a new class of edge detectors. To conclude, we are convinced that, besides the exact histogram specification, the strict ordering proposed here is a fruitful concept which, once available to the image processing community, will find a lot of interesting applications.

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Dinu Coltuc received the Diploma in electrical engineering in 1981, and the Ph.D. degree in image processing in 1996, both from the Politechnica University of Bucharest, Bucharest, Romania.

Currently, he is a Professor with Valahia University of Targoviste, Romania. He served as Visiting Professor at ESIA—University of Savoie, ENSIEG—National Polytechnic Institute of Grenoble, and Jean Monnet University of Saint Etienne, France. His research interests lie in the areas of image and signal processing and include watermarking, nonlinear processing, algorithms, and architectures.

Philippe Bolon received the engineer degree in electrical engineering in 1978, and the Ph.D. degree in signal processing in 1981, both from the National Polytechnic Institute of Grenoble, Grenoble, France.

Since 1994, he has been Professor at the School of Engineering of University of Savoie, Annecy, France. His research interests include image processing and information fusion.

Jean-Marc Chassery received the Ph.D. degree in applied mathematics from the University of Grenoble, Grenoble, France, in 1978.

He is Director of Research at CNRS, responsible for the LIS unit (Laboratoire des Images et des Signaux) englobing about 100 members, including about 30 Ph.D. students. The LIS unit is attached to CNRS, INPG, and UJF Universities at Grenoble. He develops activities around geometry for image analysis, as well as watermarking approaches to augmented-content, security, and steganalysis for images and videos.