AGENTI CHE RAGIONANO LOGICAMENTE

LOGICA FUZZY

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Fundamentals of Fuzzy Sets

- Classical set theory (1900) → crisp sets interaction. These interactions are called **operations**.
- Also fuzzy sets have well defined **properties**.
- These properties and operations are the basis on which the fuzzy sets are used to deal with uncertainty on the one hand and to represent knowledge on the other.

Definition

- **How can we represent expert knowledge that uses vague and ambiguous terms in a computer?**
- Fuzzy logic is not logic that is fuzzy, but logic that is used to describe fuzziness. Fuzzy logic is the theory of fuzzy sets, sets that calibrate vagueness.
- Fuzzy logic is based on the idea that all things admit of **degrees**. Temperature, height, speed, distance, beauty – all come on a sliding scale.
  - The motor is running **really hot**.
  - Tom is a **very tall** guy.

Definition

- Boolean logic uses sharp distinctions. It forces us to draw lines between members of a class and non-members. For instance, we may say, Tom is tall because his height is 181 cm. If we drew a line at 180 cm, we would find that David, who is 179 cm, is small.

- Is David really a small man or we have just drawn an arbitrary line in the sand?

Bit of History

- Fuzzy, or multi-valued logic, was introduced in the 1930s by Jan Lukasiewicz, who introduced logic that extended the range of truth values to all real numbers in the interval between 0 and 1.

- For example, the possibility that a man 181 cm tall is really tall might be set to a value of 0.86. It is likely that the man is tall → **possibility theory**.

- In 1965 Lotfi Zadeh paper → “Fuzzy sets”.

Why?

- Why fuzzy?

  As Zadeh said, the term is concrete, immediate and descriptive; we all know what it means.

- Why logic?

  Fuzziness rests on fuzzy set theory, and fuzzy logic is just a small part of that theory.

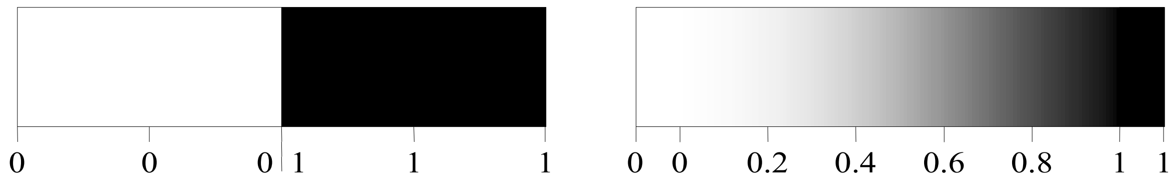
The Term "Fuzzy Logic"

- The term fuzzy logic is used in two senses:
  - **Narrow sense**: Fuzzy logic is a branch of fuzzy set theory, which deals (as logical systems do) with the representation and inference from knowledge. Fuzzy logic, unlike other logical systems, deals with imprecise or uncertain knowledge.
  - **Broad Sense**: fuzzy logic synonymously with fuzzy set theory

More Definitions

- Fuzzy logic is a set of mathematical principles for knowledge representation based on **degrees of membership**.

- Fuzzy logic uses the continuum of logical values between 0 (completely false) and 1 (completely true).



(a) Boolean Logic. (b) Multi-valued Logic.

Fuzzy Sets

- However, our own language is also the supreme expression of sets. For example, *car* indicates the *set of cars*. When we say a car, we mean one out of the set of cars.

- The classical example in fuzzy sets is tall men. The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height. (see table on next page)

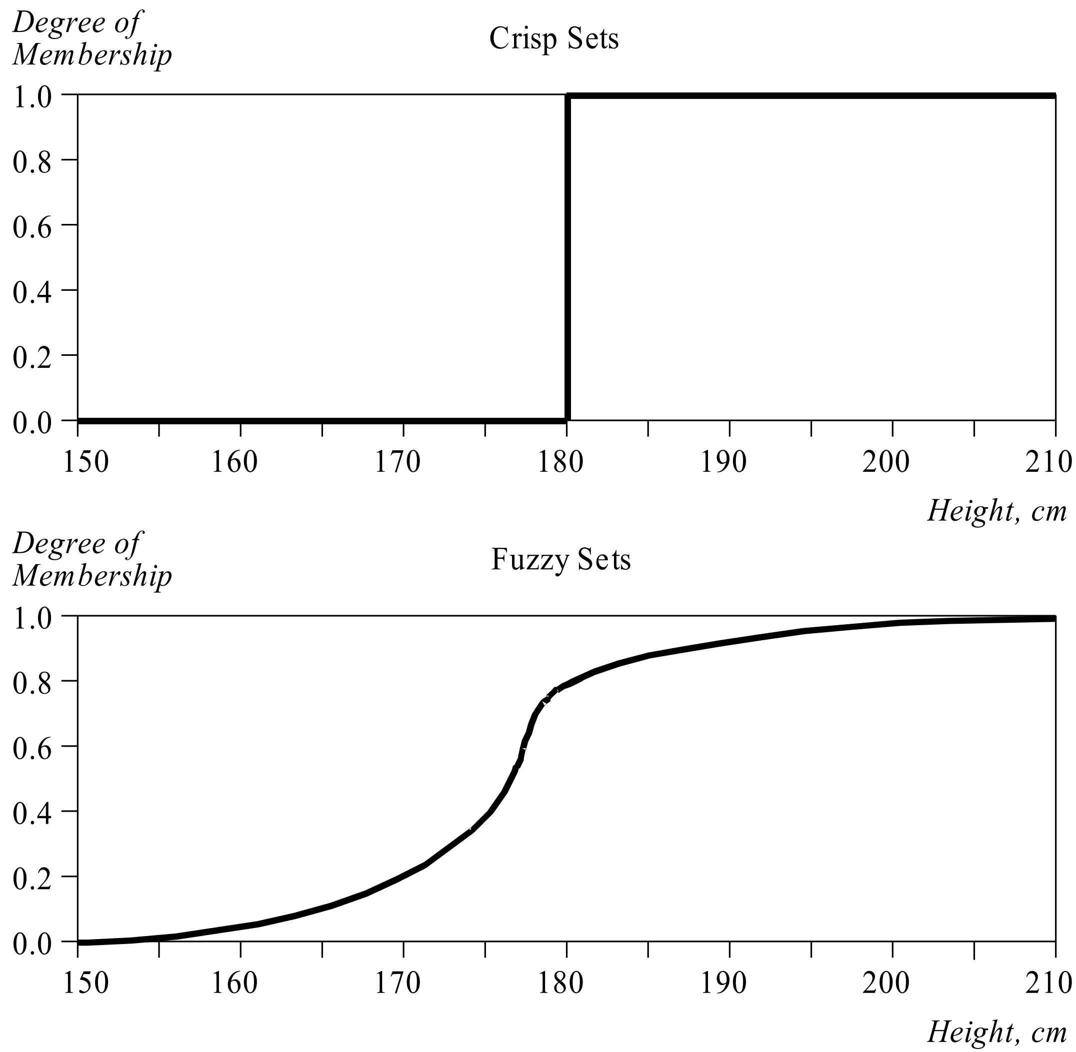
Fuzzy Sets

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

Crisp Vs Fuzzy Sets

The x-axis represents the **universe of discourse** – the range of all possible values applicable to a chosen variable.

The y-axis represents the **membership value of the fuzzy set**.



A Fuzzy Set has Fuzzy Boundaries

- Let X be the universe of discourse and its elements be denoted as x. In the classical set theory, **crisp set *A* of *X* is defined as function $f_A(x)$ called the characteristic function of *A***:

  $f_A(x) : X \rightarrow \{0, 1\}$, where

  $$f_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

  This set maps universe X to a set of two elements.

A Fuzzy Set has Fuzzy Boundaries

- In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the membership function of set A

$$\mu_A(x) : X \rightarrow \{0, 1\}, \text{ where } \begin{array}{l} \mu_A(x) = 1 \text{ if x is totally in A;} \\ \mu_A(x) = 0 \text{ if x is not in A;} \\ 0 < \mu_A(x) < 1 \text{ if x is partly in A.} \end{array}$$

- For any element *x* of universe *X*, membership function $\mu_A(x)$ equals the degree to which x is an element of set *A*. This degree, a value between 0 and 1, represents the **degree of membership**, also called **membership value**, of element *x* in set *A*.

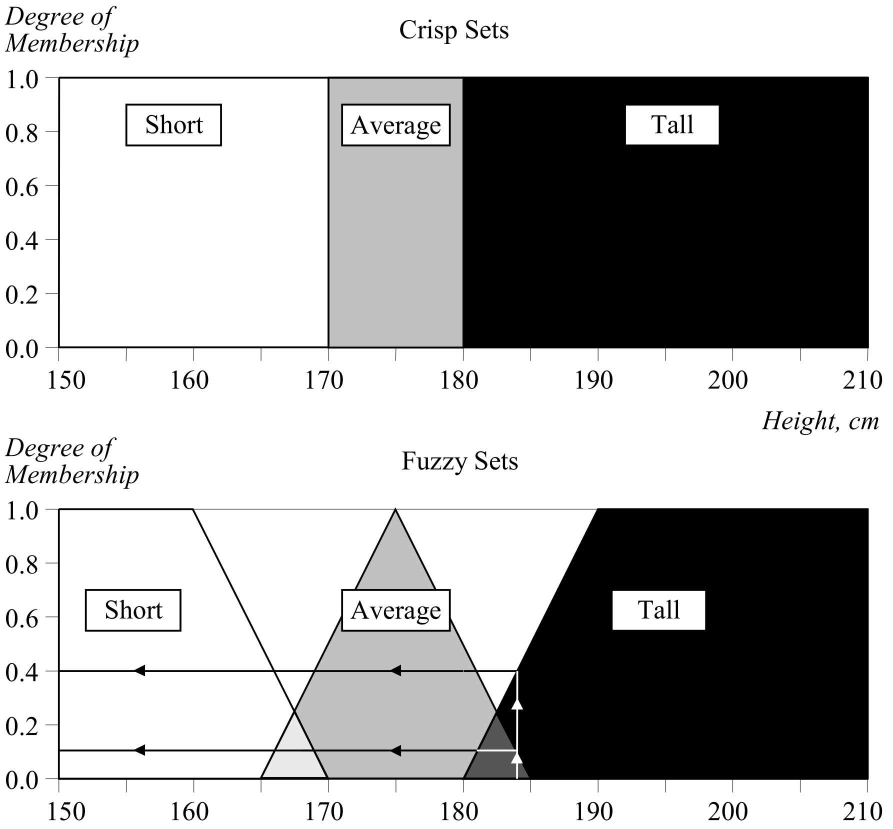
Fuzzy Set Representation

- .

- The universe of discourse – the men's heights – consists of three sets: *short*, *average* and *tall* men. As you will see, a man who is 184 cm tall is a member of the *average* men set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall* men set with a degree of 0.4. (see graph on next page)

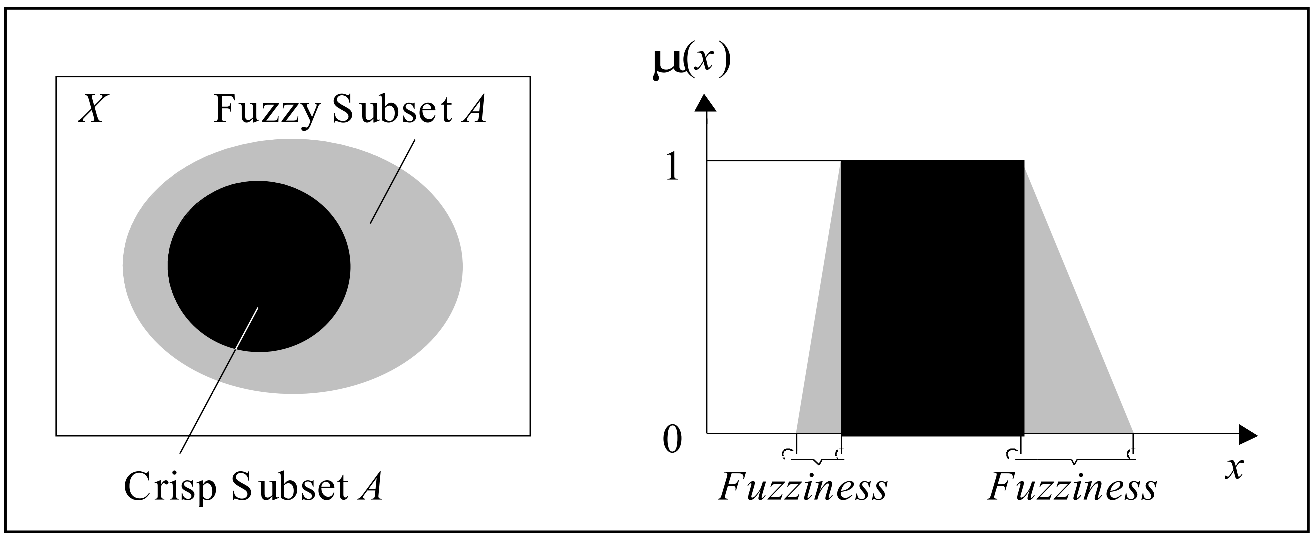
Fuzzy Set Representation

First, we determine the membership functions. In our "tall men" example, we can obtain fuzzy sets of *tall*, *short* and *average* men



Fuzzy Set Representation

- Typical functions that can be used to represent a fuzzy set are sigmoid, gaussian and pi → complex Therefore, in practice, most applications use **linear fit functions**.



Linguistic Variables and Hedges

- At the root of fuzzy set theory lies the idea of linguistic variables.
- **A linguistic variable is a fuzzy variable**. For example, the statement “John is tall” implies that the linguistic variable John takes the linguistic value tall.
- In fuzzy expert systems, linguistic variables are used in fuzzy rules. For example:

IF wind is strong
THEN sailing is good

IF project_duration is long
THEN completion_risk is high

IF speed is slow
THEN stopping_distance is short

Linguistic Variables and Hedges

- The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.

- Concept of fuzzy set qualifiers, called **hedges**.

- Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.

Membership Functions

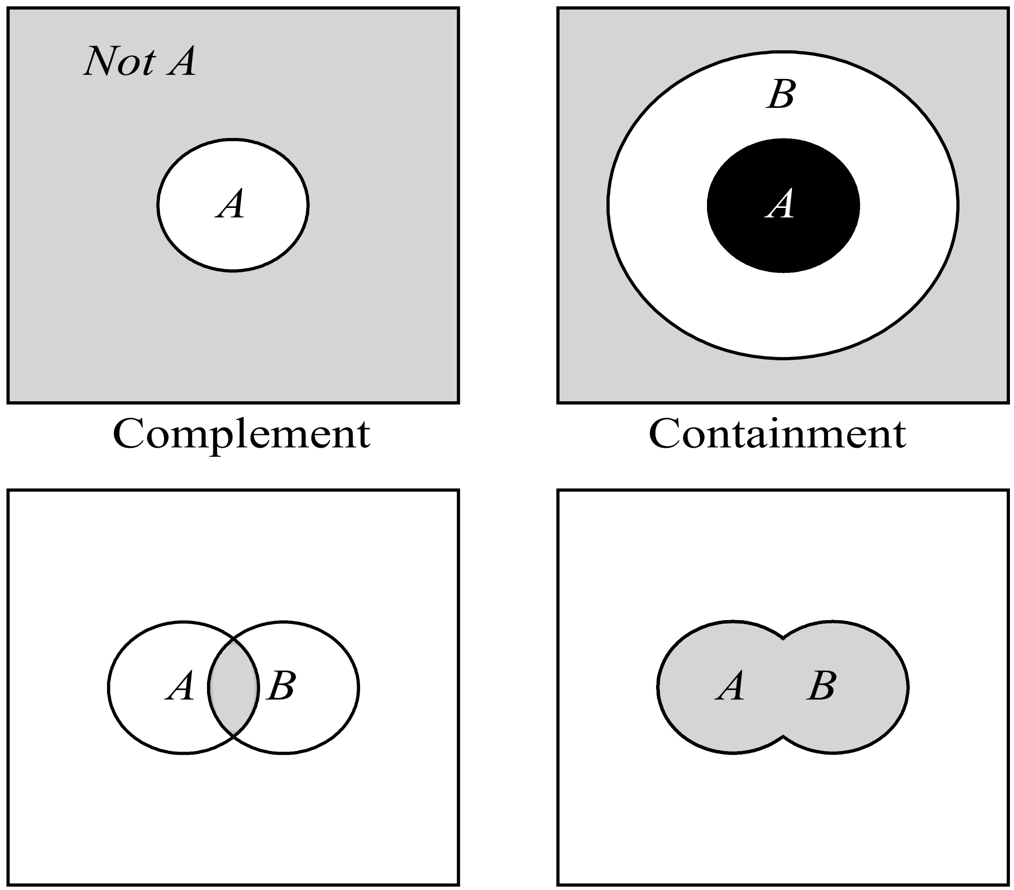
- A fuzzy set is denoted as:

$$A = \mu_A(x_i)/x_i + \ldots\ldots\ldots\ldots + \mu_A(x_n)/x_n$$

where $\mu_A(x_i)/x_i$ (a singleton) is a pair "grade of membership" element, that belongs to a finite universe of discourse:

$$A = \{x_1, x_2, .., x_n\}$$

Operations of Fuzzy Sets



Complement

Containment

Intersection

Union

Complement

- Crisp Sets: Who does not belong to the set?
- Fuzzy Sets: How much do elements not belong to the set?

- The complement of a set is an opposite of this set. For example, if we have the set of tall men, its complement is the set of NOT tall men. When we remove the tall men set from the universe of discourse, we obtain the complement.

- If A is the fuzzy set, its complement *~A* can be found as follows:

$$\mu_{\sim A}(x) = 1 - \mu_A(x)$$

Containment

- Crisp Sets: Which sets belong to which other sets?
- Fuzzy Sets: Which sets belong to other sets?

- A set can contain other sets. The smaller set is called the **subset**. Example:
  - the set of tall men contains all tall men; very tall men is a subset of tall men. However, the tall men set is just a subset of the set of men.
  - In crisp sets, all elements of a subset entirely belong to a larger set.
  - In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in it than in the larger set.

Intersection

- Crisp Sets: Which element belongs to both sets?
- Fuzzy Sets: How much of the element is in both sets?

- In classical set theory, an intersection between two sets contains the elements shared by these sets. In fuzzy sets, an element may partly belong to both sets with different memberships.

- A fuzzy intersection is the **lower membership** in both sets of each element. The fuzzy intersection of two fuzzy sets *A* and *B* on universe of discourse X:

$$\mu_{A \cap B}(x) = \min\,[\mu_A(x), \mu_B(x)] = \mu_A(x) \cap \mu_B(x),$$

where $x \in X$

Union

- <u>Crisp Sets</u>: Which element belongs to either set?
- <u>Fuzzy Sets</u>: How much of the element is in either set?

- The union of two crisp sets consists of every element that falls into either set.

- In fuzzy sets, the union is the reverse of the intersection. That is, the union is the **largest membership** value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:

$$\mu A \cup B(x) = \max[\mu A(x), \mu B(x)] = \mu A(x) \cup \mu B(x),$$

where $x \in X$

Properties of Fuzzy Sets : Equality

- Fuzzy set A is considered equal to a fuzzy set B, IF AND ONLY IF (iff):

$$\mu_A(x) = \mu_B(x),\ \forall x \in X$$

$A = 0.3/1 + 0.5/2 + 1/3$

$B = 0.3/1 + 0.5/2 + 1/3$

therefore $A = B$

Properties of Fuzzy Sets: Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

  $$\mu_A(x) \leq \mu_B(x), \ \forall x \in X$$

  Consider $X = \{1, 2, 3\}$ and sets A and B

  $A = 0.3/1 + 0.5/2 + 1/3;$

  $B = 0.5/1 + 0.55/2 + 1/3$

  then A is a subset of B, or $A \subseteq B$

Properties of Fuzzy Sets: Cardinality

- Cardinality of a non-fuzzy set, Z, is the number of elements in Z. BUT the cardinality of a fuzzy set A is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

  $$card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \Sigma\mu_A(x_i), \qquad \text{for } i=1..n$$

  Consider $X = \{1, 2, 3\}$ and sets A and B

  $A = 0.3/1 + 0.5/2 + 1/3;$

  $B = 0.5/1 + 0.55/2 + 1/3$

  $card_A = 1.8$

  $card_B = 2.05$

Properties of Fuzzy Sets: Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:

  $\mu_A(x) = 0, \forall x \in X$

  Consider $X = \{1, 2, 3\}$ and set A

  $A = 0/1 + 0/2 + 0/3$

  then A is *empty*

Fuzzy Set Normality

- A fuzzy subset of X is called **normal** if there exists at least one element $x \in X$ such that $\mu_A(x) = 1$.
- A fuzzy subset that is not normal is called **subnormal**.
- All crisp subsets except for the null set are normal. In fuzzy set theory, the concept of nullness essentially generalises to subnormality.
- The **height** of a fuzzy subset *A* is the large membership grade of an element in *A*

$$height(A) = \max_x(\mu_A(x))$$

Fuzzy Sets Core and Support

- Assume A is a fuzzy subset of X:
- the **support** of A is the crisp subset of X consisting of all elements with membership grade:

  $$supp(A) = \{x \mid \mu_A(x) > 0 \text{ and } x \in X\}$$

- the **core** of A is the crisp subset of X consisting of all elements with membership grade:

  $$core(A) = \{x \mid \mu_A(x) = 1 \text{ and } x \in X\}$$

Fuzzy Set Math Operations

- $aA = \{a\mu_A(x), \forall x \in X\}$

  Let a =0.5, and

  A = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  aA = {0.25/a, 0.15/b, 0.1/c, 0.5/d}

- $A^a = \{\mu_A(x)^a, \forall x \in X\}$

  Let a =2, and

  A = {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  A^a = {0.25/a, 0.09/b, 0.04/c, 1/d}

- …

Fuzzy Sets Examples

- Consider two fuzzy subsets of the set X,

  X = {a, b, c, d, e }

  referred to as A and B

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  and

  B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

Fuzzy Sets Examples

- <u>Support</u>:

  $supp(A)$ = {a, b, c, d }

  $supp(B)$ = {a, b, c, d, e }

- <u>Core</u>:

  $core(A)$ = {a}

  $core(B)$ = {o}

- <u>Cardinality</u>:

  $card(A) = 1+0.3+0.2+0.8+0 = 2.3$

  $card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$

Fuzzy Sets Examples

- <u>Complement</u>:

  A = {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  $\neg A$ = {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

- <u>Union</u>:

  $A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

- <u>Intersection</u>:

  $A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

  Recall B = {0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e}

Fuzzy Rules

- A fuzzy rule can be defined as a conditional statement in the form:

IF	*x*	is *A*
THEN	*y*	is *B*

- where *x* and *y* are linguistic variables; and *A* and *B* are linguistic values determined by fuzzy sets on the universe of discourses *X* and *Y*, respectively.

Classical Vs Fuzzy Rules

- A classical IF-THEN rule uses binary logic, for example,

<u>Rule: 1</u>
IF speed is > 100
THEN stopping_distance is long

<u>Rule: 2</u>
IF speed is < 40
THEN stopping_distance is short

- The variable speed can have any numerical value between 0 and 220 km/h, but the linguistic variable stopping_distance can take either value long or short. In other words, classical rules are expressed in the black-and-white language of Boolean logic.

Classical Vs Fuzzy Rules

- We can also represent the stopping distance rules in a fuzzy form:

  Rule: 1
  IF speed is fast
  THEN stopping_distance is long

  Rule: 2
  IF speed is slow
  THEN stopping_distance is short

- In fuzzy rules, the linguistic variable speed also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as slow, medium and fast.

Classical Vs Fuzzy Rules

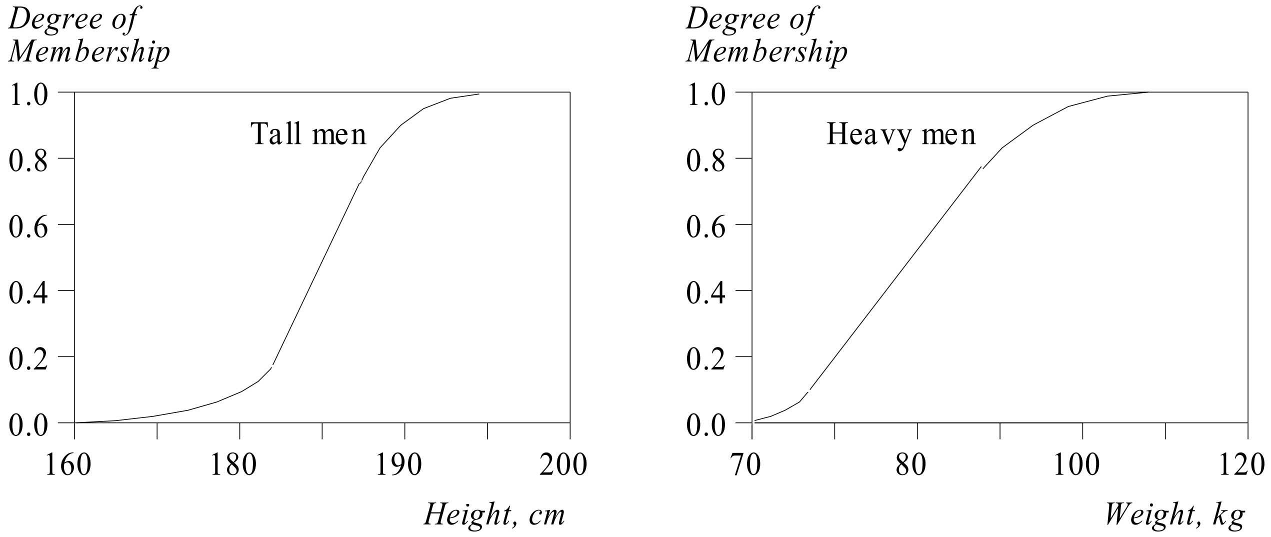
- Fuzzy rules relate fuzzy sets.
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

Firing Fuzzy Rules

- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

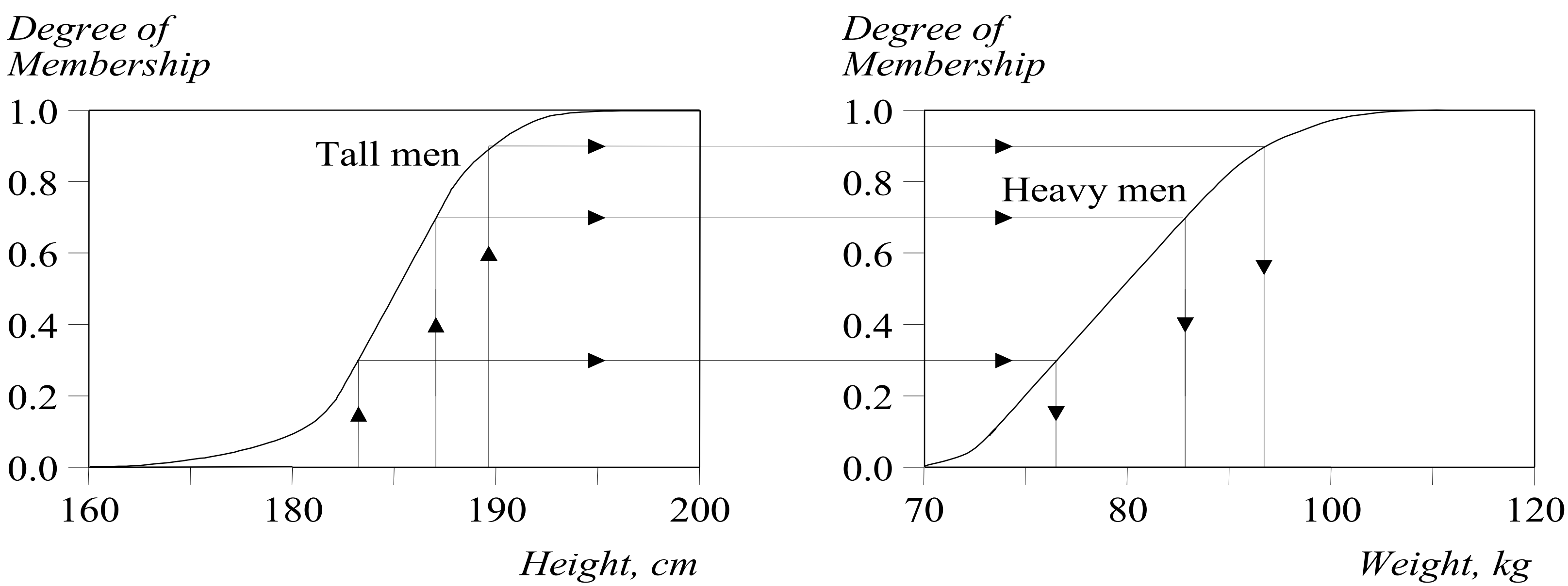
  IF height is tall

  THEN weight is heavy



Firing Fuzzy Rules

- The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection.**



Firing Fuzzy Rules

- A fuzzy rule can have multiple antecedents, for example:

  IF project_duration is long
  AND project_staffing is large
  AND project_funding is inadequate
  THEN risk is high

  IF service is excellent
  OR food is delicious
  THEN tip is generous

- The consequent of a fuzzy rule can also include multiple parts, for instance:

  IF temperature is hot
  THEN hot_water is reduced;
  cold_water is increased

Fuzzy Sets Example

- Air-conditioning involves the delivery of air which can be warmed or cooled and have its humidity raised or lowered.
- An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has cooler and the cooler is under thermostatic control. Generally, the amount of air being compressed is proportional to the ambient temperature.
- Consider Johnny's air-conditioner which has five control switches: COLD, COOL, PLEASANT, WARM and HOT. The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: MINIMAL, SLOW, MEDIUM, FAST and BLAST.

Fuzzy Sets Example

- The rules governing the air-conditioner are as follows:

RULE 1:
IF TEMP is COLD THEN SPEED is MINIMAL

RULE 2:
IF TEMP is COOL THEN SPEED is SLOW

RULE 3:
IF TEMP is PLEASANT THEN SPEED is MEDIUM

RULE 4:
IF TEMP is WARM THEN SPEED is FAST

RULE 5:
IF TEMP is HOT THEN SPEED is BLAST

Fuzzy Sets Example

The **temperature** graduations are related to Johnny's perception of ambient temperatures.

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Temp (^{0}C).	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
15	N	Y	N	N	N
17.5	N	N	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

Fuzzy Sets Example

Johnny's perception of the **speed** of the motor is as follows:

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Rev/sec (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:
- COLD:

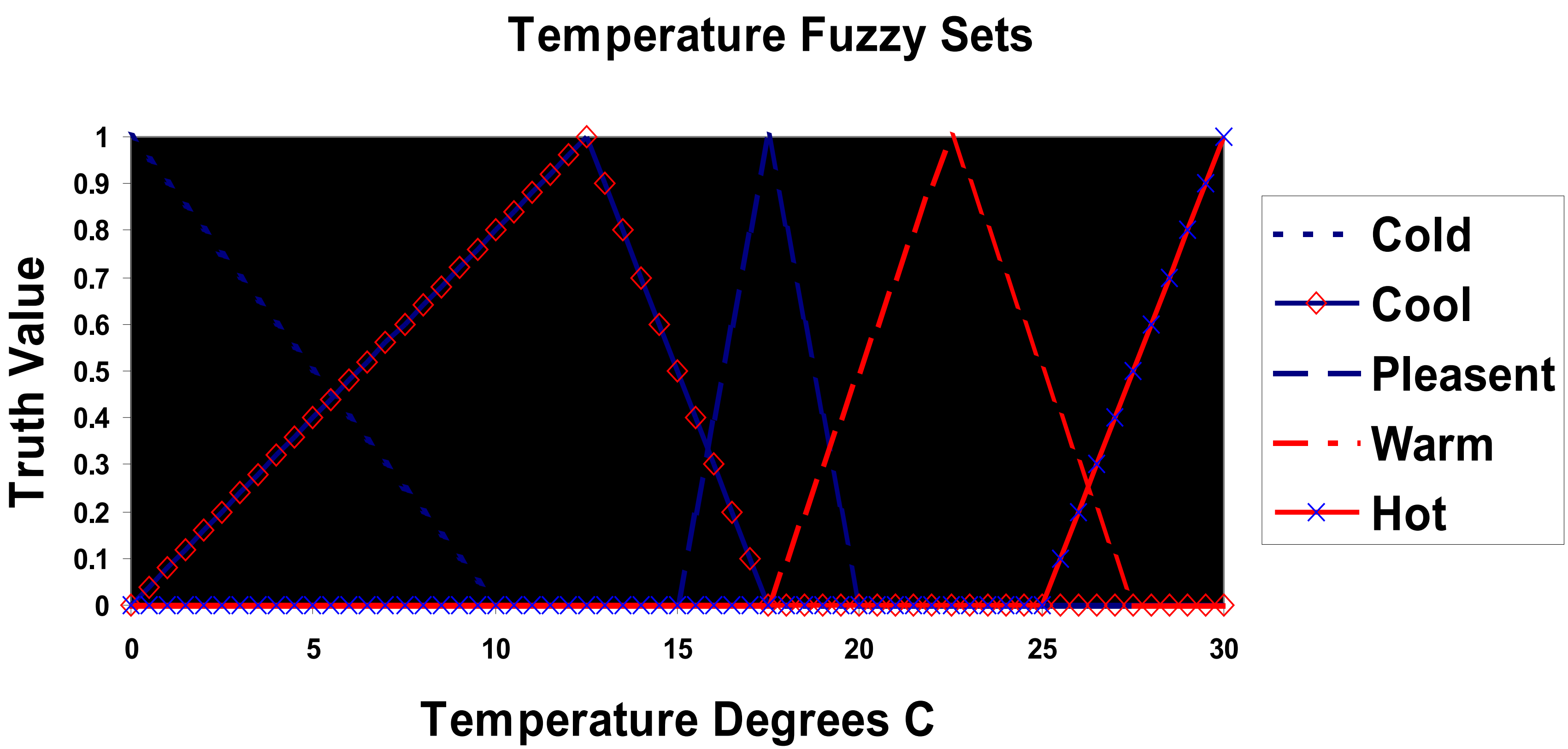
  for $0 \leq t \leq 10$ $\mu_{COLD}(t) = -t / 10 + 1$
- SLOW:

  for $0 \leq t \leq 12.5$ $\mu_{SLOW}(t) = t / 12.5$

  for $12.5 \leq t \leq 17.5$ $\mu_{SLOW}(t) = -t / 5 + 3.5$
- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:

- MINIMAL:

  for $0 \leq v \leq 30$ $\mu_{COLD}(t) = -v / 30 + 1$

- SLOW:

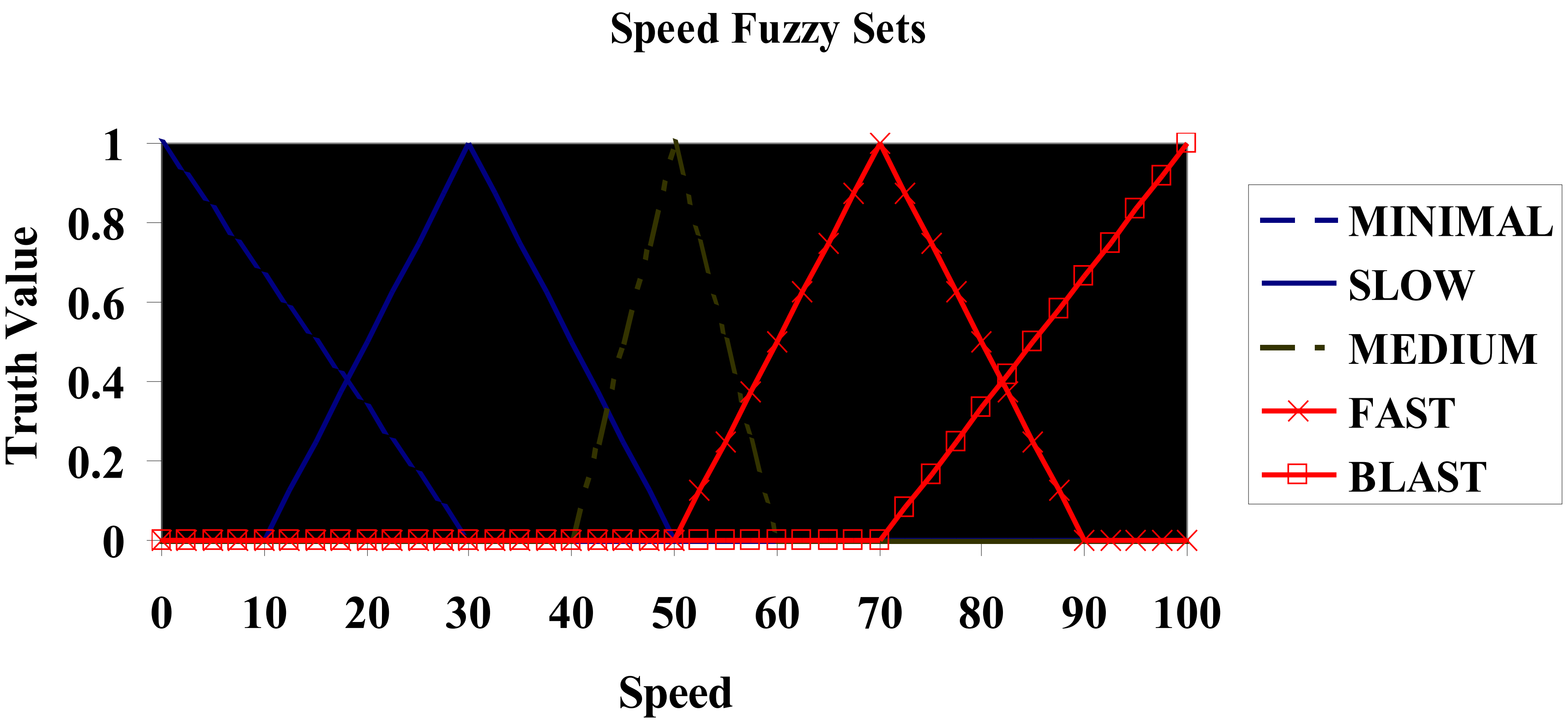
  for $10 \leq v \leq 30$ $\mu_{SLOW}(t) = v / 20 - 0.5$

  for $30 \leq v \leq 50$ $\mu_{SLOW}(t) = -v / 20 + 2.5$

- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Exercises

For

$A = \{0.2/a, 0.4/b, 1/c, 0.8/d, 0/e\}$

$B = \{0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e\}$

Then, calculate the following:

- Support, Core, Cardinality, and Complement for *A* and *B* independently
- Union and Intersection of A and B
- the new set *C*, if $C = A^2$
- the new set *D*, if *D* = 0.5□*B*
-

Solutions

A = {0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}
B = {0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

<u>Support</u>

Supp(A) = {a, b, c, d}
Supp(B) = {b, c, d, e}

<u>Core</u>

Core(A) = {c}
Core(B) = {}

<u>Cardinality</u>

Card(A) = 0.2 + 0.4 + 1 + 0.8 + 0 = 2.4
Card(B) = 0 + 0.9 + 0.3 + 0.2 + 0.1 = 1.5

<u>Complement</u>

Comp(A) = {0.8/a, 0.6/b, 0/c, 0.2/d, 1/e}
Comp(B) = {1/a, 0.1/b, 0.7/c, 0.8/d, 0.9/e}

Solutions

A = {0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}
B = {0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

<u>Union</u>

A□B = {0.2/a, 0.9/b, 1/c, 0.8/d, 0.1/e}

<u>Intersection</u>

A□B = {0/a, 0.4/b, 0.3/c, 0.2/d, 0/e}

<u>C=A^2</u>

C = {0.04/a, 0.16/b, 1/c, 0.64/d, 0/e}

<u>D = 0.5□B</u>

D = {0/a, 0.45/b, 0.15/c, 0.1/d, 0.05/e}

Fuzzy Inference

- The most commonly used fuzzy inference technique is the so-called **Mamdani** method.

- In 1975, Professor Ebrahim Mamdani of London University built one of the first fuzzy systems to control a steam engine and boiler combination. He applied a set of fuzzy rules supplied by experienced human operators.

Mamdani Fuzzy Inference

- The Mamdani-style fuzzy inference process is performed in four steps:

1. Fuzzification of the input variables

1. Rule evaluation (inference)

1. Aggregation of the rule outputs (composition)

1. Defuzzification.

Mamdani Fuzzy Inference

We examine a simple two-input one-output problem that includes three rules:

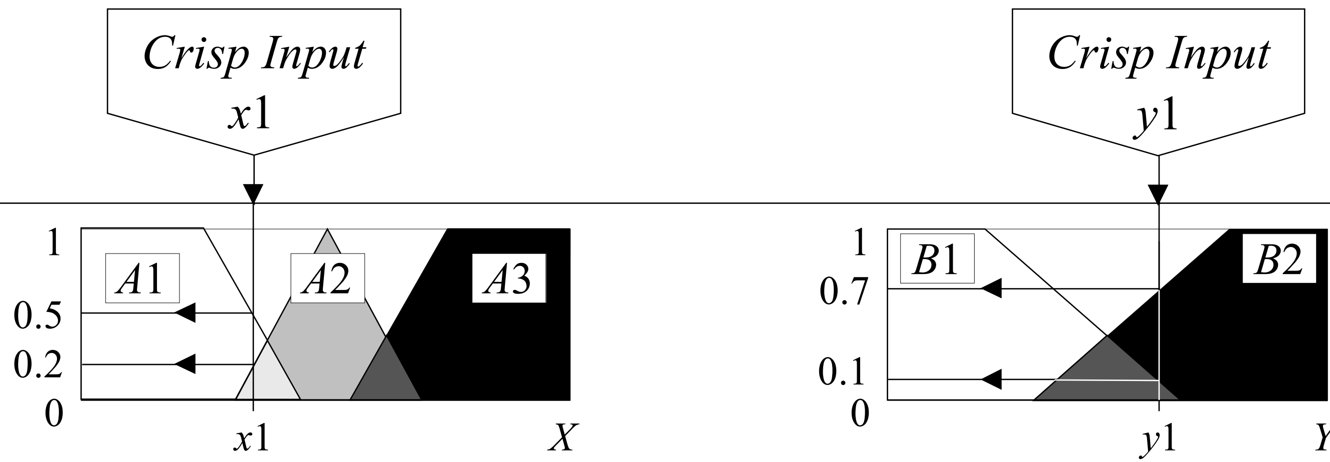
Rule: 1		Rule: 1		
IF	x is A3	IF	project_funding	is adequate
OR	y is B1	OR	project_staffing	is small
THEN	z is C1	THEN	risk	is low

Rule: 2		Rule: 2		
IF	x is A2	IF	project_funding	is marginal
AND	y is B2	AND	project_staffing	is large
THEN	z is C2	THEN	risk	is normal

Rule: 3		Rule: 3		
IF	x is A1	IF	project_funding	is inadequate
THEN	z is C3	THEN	risk	is high

Step 1: Fuzzification

- The first step is to take the crisp inputs, x1 and y1 (*project funding* and *project staffing*), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



$$\mu_{(x=A1)} = 0.5$$
$$\mu_{(x=A2)} = 0.2$$
$$\mu_{(y=B1)} = 0.1$$
$$\mu_{(y=B2)} = 0.7$$

Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.

- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.

- This number (the truth value) is then applied to the consequent membership function.

Step 2: Rule Evaluation

RECAL:

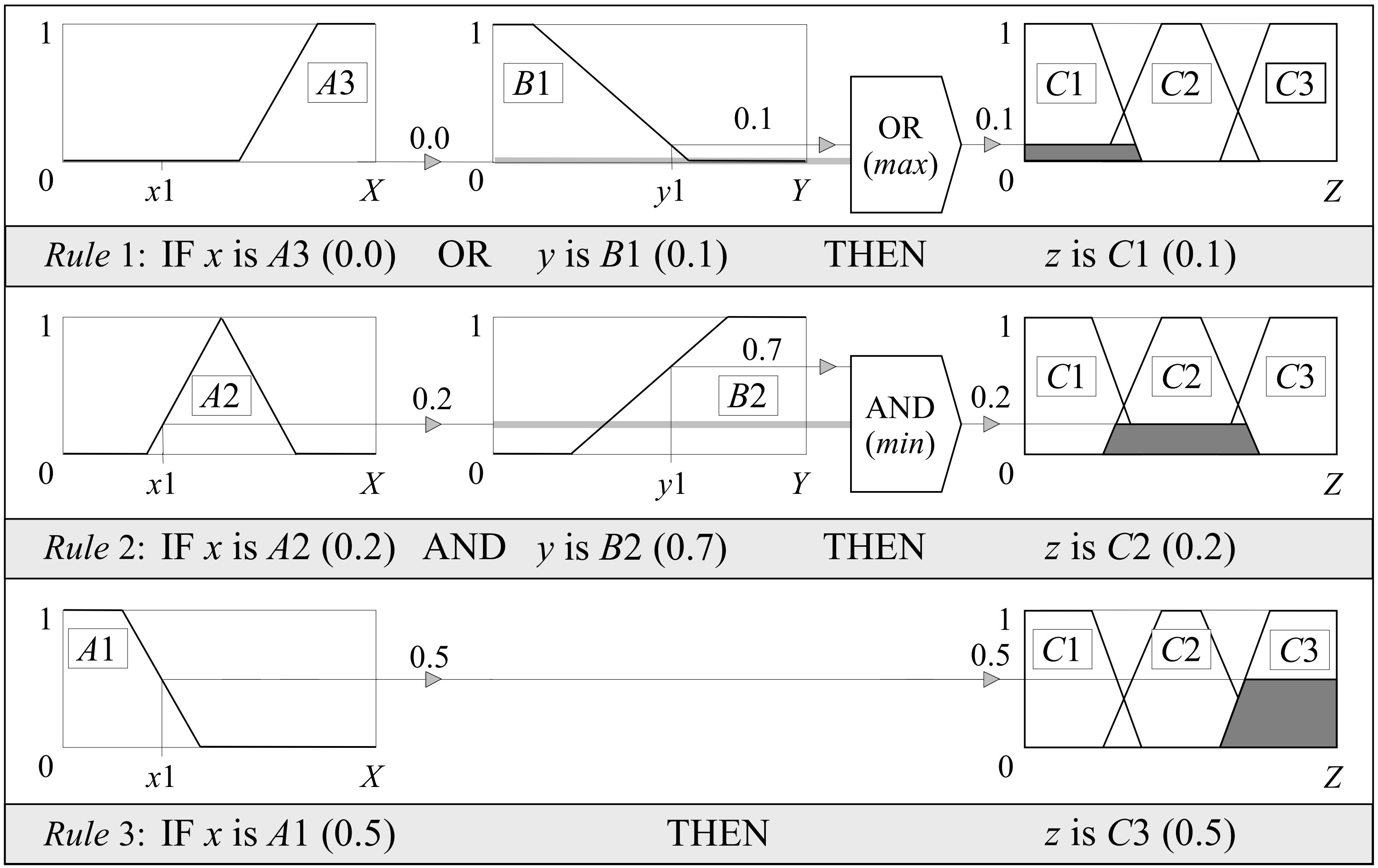
To evaluate the disjunction of the rule antecedents, we use the OR fuzzy operation. Typically, fuzzy expert systems make use of the classical fuzzy operation union:

$$\mu_{A \cup B}(x) = \max\,[\mu_A(x),\ \mu_B(x)]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the AND fuzzy operation intersection:

$$\mu_{A \cap B}(x) = \min\,[\mu_A(x),\ \mu_B(x)]$$

Step 2: Rule Evaluation



Step 2: Rule Evaluation

- Now the result of the antecedent evaluation can be applied to the membership function of the consequent.

- There are two main methods for doing so:
  - Clipping
  - Scaling

Step 2: Rule Evaluation

- The most common method of correlating the rule consequent with the truth value of the rule antecedent is to cut the consequent membership function at the level of the antecedent truth. This method is called **clipping** (alpha-cut).

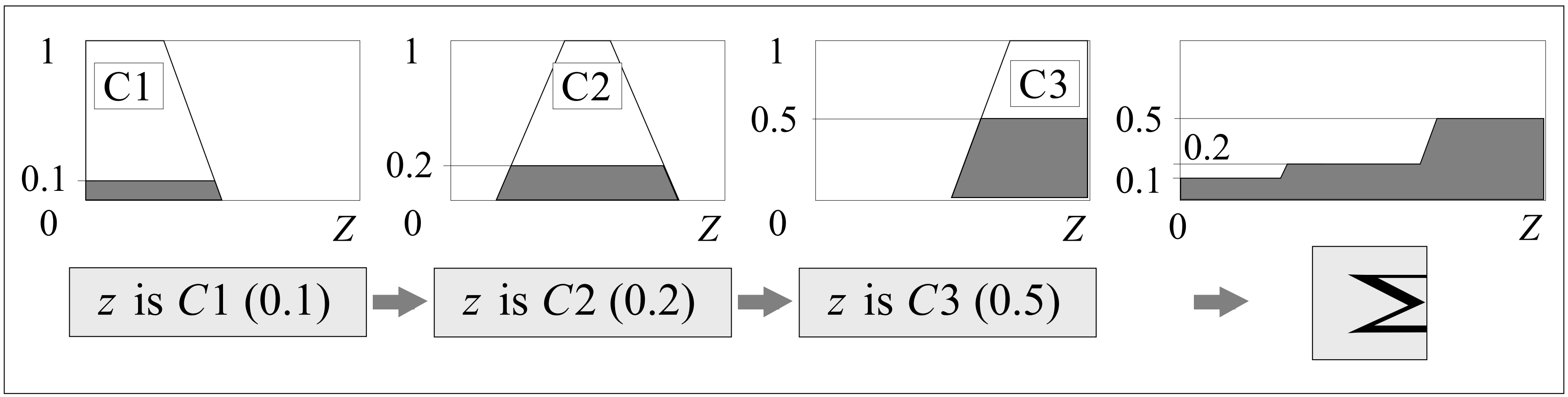
- Since the top of the membership function is sliced, the clipped fuzzy set loses some information.

- However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.

Step 3: Aggregation of the rule outputs

- Aggregation is the process of unification of the outputs of all rules.
- We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.
- The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.

Step 3: Aggregation of the rule outputs



Step 4: Defuzzification

- The last step in the fuzzy inference process is defuzzification.
- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.

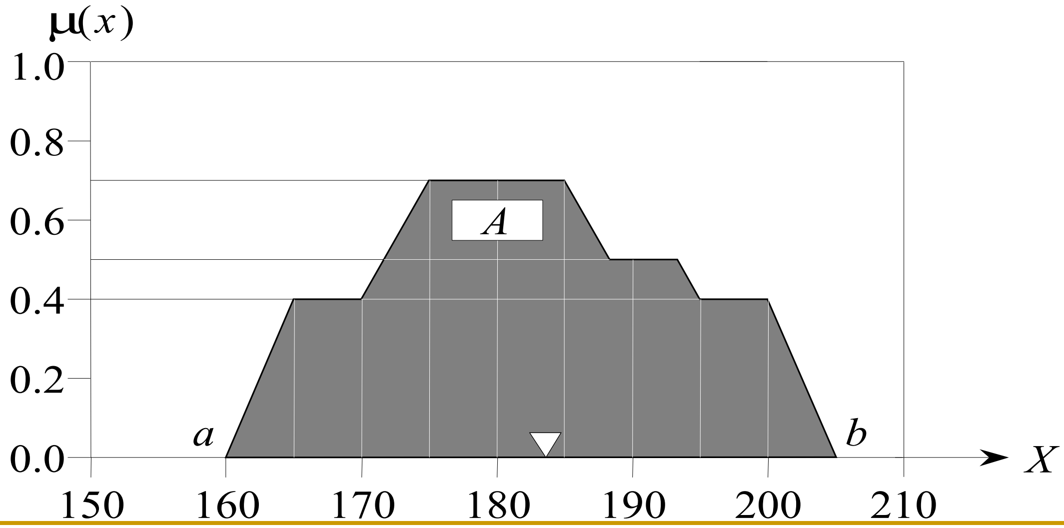
Step 4: Defuzzification

- There are several defuzzification methods, but probably the most popular one is the **centroid technique**. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity** (**COG**) can be expressed as:

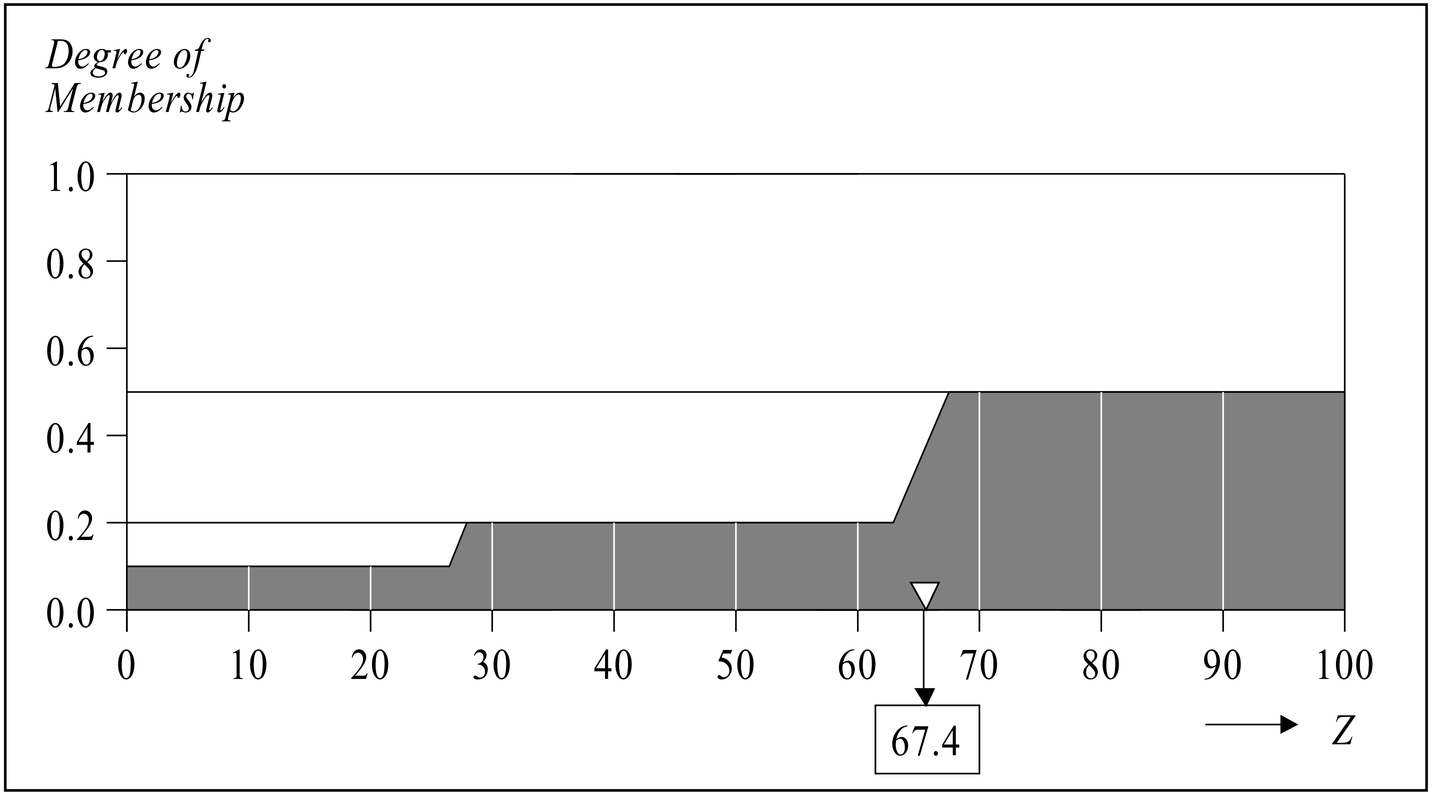
$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

Step 4: Defuzzification

- Centroid defuzzification method finds a point representing the centre of gravity of the fuzzy set, *A*, on the interval, *ab*.
- A reasonable estimate can be obtained by calculating it over a sample of points.



Step 4: Defuzzification



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$

Sugeno Fuzzy Inference

- Michio Sugeno suggested to use a single spike, a singleton, as the membership function of the rule consequent.

- A singleton, or more precisely a fuzzy singleton, is a fuzzy set with a membership function that is unity at a single particular point on the universe of discourse and zero everywhere else.

Sugeno Fuzzy Inference

- Sugeno-style fuzzy inference is very similar to the Mamdani method. Sugeno changed only a rule consequent. Instead of a fuzzy set, he used a mathematical function of the input variable. The format of the **Sugeno-style fuzzy rule** is

  IF x is A
  AND y is B
  THEN z is f(x, y)

  where x, y and z are linguistic variables; A and B are fuzzy sets on universe of discourses X and Y, respectively; and f(x, y) is a mathematical function.

Sugeno Fuzzy Inference

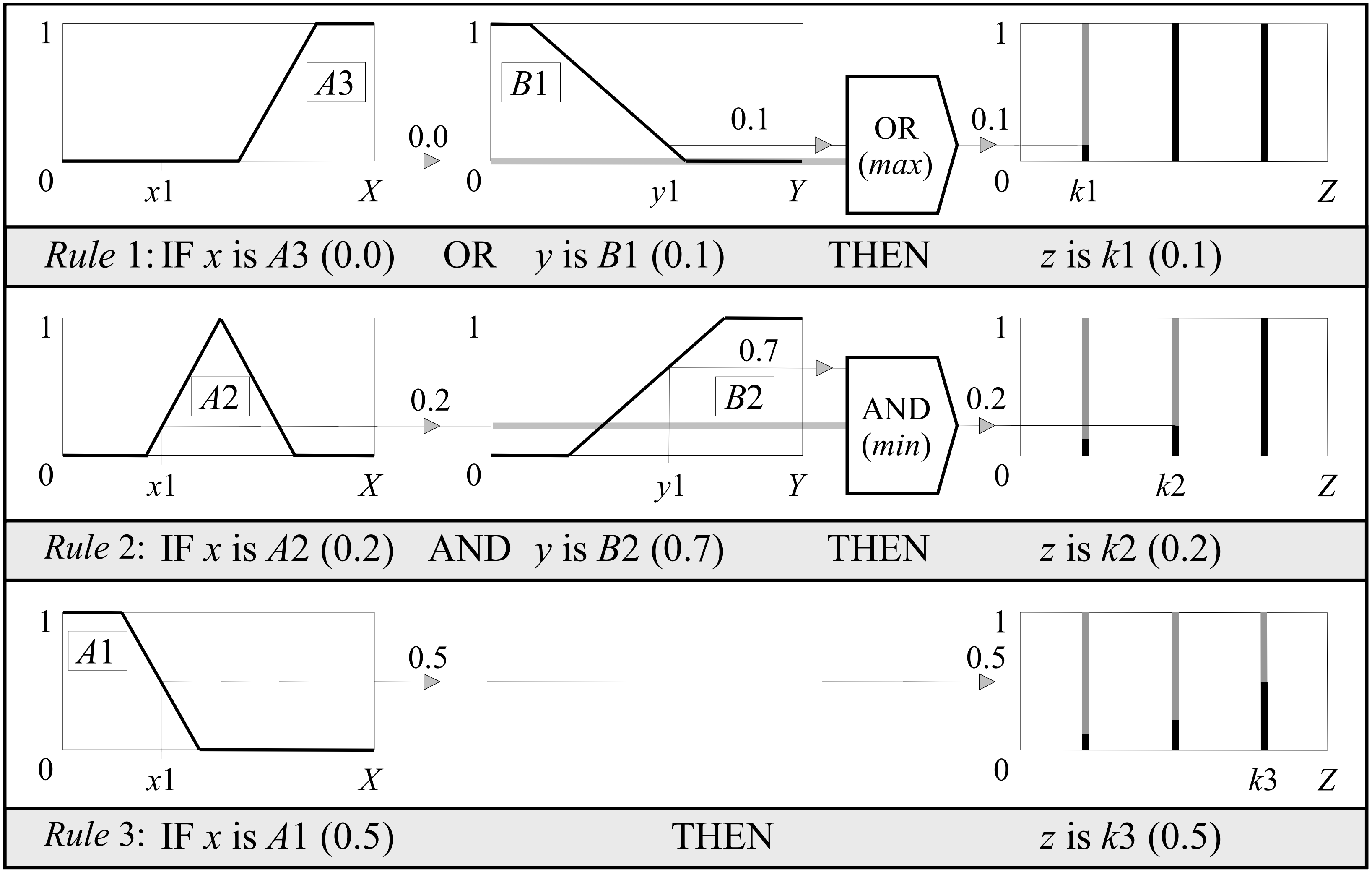
- The most commonly used **zero-order Sugeno fuzzy model** applies fuzzy rules in the following form:

  IF $\quad x$ is A
  AND $\quad y$ is B
  THEN z is k

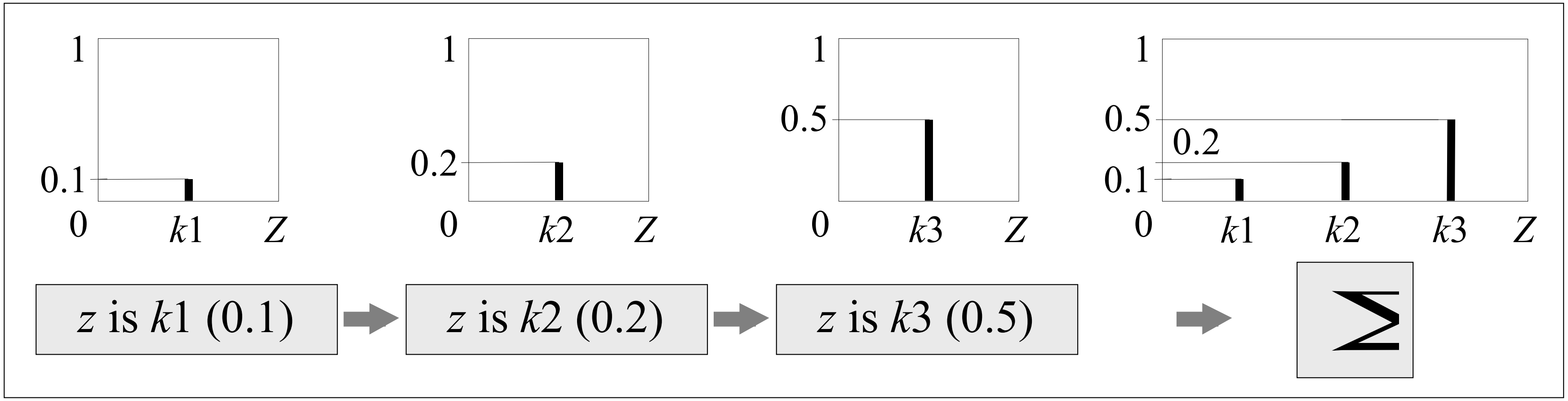
  where k is a constant.

- In this case, the output of each fuzzy rule is constant. All consequent membership functions are represented by singleton spikes.

Sugeno Rule Evaluation



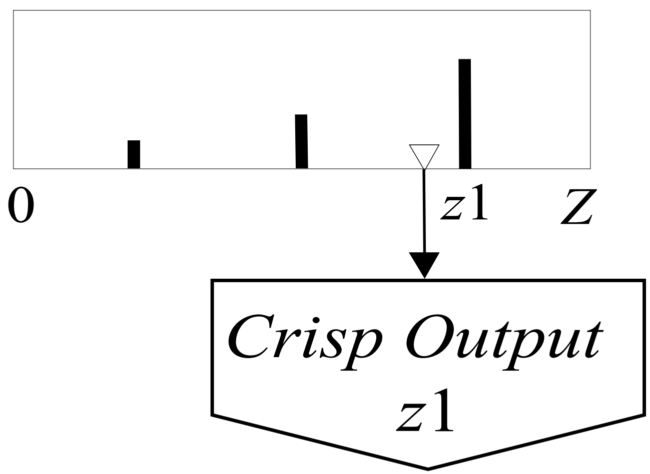
Sugeno Aggregation of the Rule Outputs



Sugeno Defuzzification

Weighted Average (WA)

$$WA = \frac{\mu(k1) \times k1 + \mu(k2) \times k2 + \mu(k3) \times k3}{\mu(k1) + \mu(k2) + \mu(k3)} = \frac{0.1 \times 20 + 0.2 \times 50 + 0.5 \times 80}{0.1 + 0.2 + 0.5} = 65$$



Mamdani or Sugeno?

- Mamdani method is widely accepted for capturing expert knowledge. It allows us to describe the expertise in more intuitive, more human-like manner. However, Mamdani-type fuzzy inference entails a substantial computational burden.

- On the other hand, Sugeno method is computationally effective and works well with optimisation and adaptive techniques, which makes it very attractive in control problems, particularly for dynamic nonlinear systems.