L&PS – Logic and Philosophy of Science
Vol. IX, No. 1, 2011, pp. 131-150

The "enjeux" of the mathematical infinite

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> […] mathematics is the science of the infinite. It was the great achievement of the Greeks to make the tension between the finite and infinite fruitful for the analysis of reality.
>
> Hermann Weyl

ABSTRACT. The present contribution, on the one hand, tries to bring out the nature of certain changes of the idea of infinite that occurred within some areas of the logical-mathematical thought during the first half of the XXth century. The critical use of certain contributions taken from the French epistemology, was applied in order to achieve a conceptual-historical significance of the shift from a "static" to a 'dynamic' view of the infinite, which is shown in the work on linear logic by Jean-Yves Girard. On the other hand, however, it will show the more relevant issues that the use of infinite has revealed within the field of mathematical research at the end of the XXth century.

1.

The scientific and philosophical thought of the XXth century, as is common knowledge, has led to an enormous amount of critical reflection on the structure of science, in order to speak about a full-blown "epistemological heritage", an "epistemological-hermeneutic arsenal", an "epistemological rea-

Published by L&PS – Logic and Philosophy of Science
http://www2.units.it/ ~episteme — ISSN: 1826-1043

son"[1]. The conceptual tools are formed to allow the philosophy of science to go into what Gaston Bachelard called "la pensée des sciences"[2], in order to bring out the size and joints of the "enjeux"[3] of "theorein". In a particular way, the secular-long effort for reflective criticism has focused on the nature of mathematics as a particular form of knowledge, on the specificity of concepts and instruments used with an heuristic value. Among these ones, it comes as no surprise that the issue of infinite has been identified for its nature of "concept à entrees multiples"[4], for the particular role played in the specific mathematical *theorein*; wherein, from the beginning it has enabled the birth and development of different maturing sectors throughout the centuries. That is why, on the wake of Henri Poincaré, it is not pointless to speak about a "logic of the infinite", that is, a logic of the discovery of several otherwise inaccessible routes, for it has been and is up to this day, an instrument of reflection. The infinite, in fact, provides access to several research areas such as a bearer of increasingly complex concepts and its continuing and sometimes obscure "enjeux" are characterized as more and more effective means of understanding. From the beginning it constituted one of those particular "sub-soils" in the math world, to use a beautiful expression by Jean Desanti[5], which should be investigated to have a more unified view of the infinite. It is at the bottom of the "expérience mathématique"[6] and it allows us to understand the mathematical world as a living organism, as *unicum*, which has already been stressed by Federigo Enriques in *I Problemi della scienza*, 1906: handling the infinite has meant and still means not being able to take a part of the mathe-

[1] On the "epistemological-hermeneutic arsenal" speaks ANTISERI 2000, pg. 226; on the "epistemological reason", SALANSKIS 1997, pg. 414 and FRIEDMAN 2006, pgs. 149-160; on "epistemological heritage", see CASTELLANA 1990 and 2004, *passim*.

[2] In *La philosophie du non* of 1940, Gaston Bachelard had already spoken about "pensée des sciences", of implicit philosophical dimension in scientific act (Bachelard 1969, ch. I). But already in 1912 in *Scienza e Razionalismo*, the Italian mathematician Federigo Enriques had boosted the concept of 'implicit philosophy' in science. (ENRIQUES 1987, pg. 112).

[3] We take the idea of 'enjeux' from CHÂTELET 1993; on this concept see CASTELLANA 2004, chap. VI.

[4] DESANTI 1975, pg. 264; on Desanti, see CASTELLANA 1985, chap. II.

[5] DESANTI 1968, pg.

[6] CAVAILLÈS-LAUTMAN 1946 and see CASTELLANA 1990.

matical world and detach it from anything in so far as it is a resource that allows to continue living under the condition of preserving its own integrity.
The mathematical experience of the infinite allows us to identify new pathways and, to quote the French mathematician Alain Connes, it outlines:

> "the long *periplus* around which a mathematician travels" marks the stages of "a journey in another geography, in another territory, in the course in which he is faced with another reality. This mathematical reality is equally complex, equally obscure form the material reality in which we live in "[7].

The continuous and constant research on the infinite have traced the *periploi* of always newer territories, such as to mark both historically and conceptually, what Leonardo da Vinci called "le infinite ragioni del reale silente" ["the infinite reasons for the silent reality"], this has allowed to move from one "realm" to another inside the mathematical world, to use an effective expression made by Hermann Weyl. At the same time, it has opened important new paths for scientific reason in other fields such as physics, in such a way that it can be defined, using another expression by Gilles Châtelet, as a full-blown scientific "mobile" for its ability to take shape in disciplines such as mathematics, logic, physics and to guide their own pathways. Its ability to expand the boundaries of the individual sciences and continuously open new horizons for research, explains the fact that it has been and still is, the focus of continuous discussions of an epistemological nature. Quoting Federigo Enriques a mathematician who has dedicated many studies of a historical-conceptual character to Greek scientific thought, the infinite is the creative tool par excellence, paving the way for "mathematical poetry"[8] because of its intrinsic complexity. It is the basis of that single event that Michel Serres, in his studies on the origin of geometry, has called "the Greek miracle"[9], which consists precisely on the birth of mathematics. If you investigate, if you question, if you clarify the problem of the infinite, continues Serres, you investigate, you

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7 CONNES 2007, pg. 55.

8 See ENRIQUES 1922, pg. 19 e 2007, pg. 25; like this see CASTELLANA 2007, pgs. 87-127.

9 See SERRES 1989 e 1993; on this see CASTELLANA 2001, pg. 209-224.

question, you consult the alternating and contradictory happenings of our rationality along with the current outcomes; the constant historical and epistemological reflection has rightly focused on its being a "logos alogos"[10], just as very often Simone Weil underlines in numerous fragments in his analysis of Platonic thought.

Therefore, from the beginning, the idea of the infinite has involved more areas of scientific thought, where it has assumed various conceptual features and has triggered an always more complex series of researches. It becomes useful as a background for several areas of scientific rationality, as Enriques says, so that you can ascertain that its presence or absence can function as a line of conceptual demarcation and delimitation between various areas in human thinking[11]. The historical and epistemological analysis of the research fields that have incorporated this idea, as *in primis*, the mathematical and the logical thought, gives way to get and dispose of a wide range of theoretical perspectives usable in other contexts and specific applications. Given that the infinite is a "logos alogos", a "concept à entrees multiples" and multi-dimensional has allowed itself to range freely in disparate domains and free them from essentialist assumptions: this has been the case for logical thinking, which has gradually made some of its basic principles increasingly "souples", that is, operational and open to the world. Therefore, most of the history of twentieth century logical-mathematical thinking can be interpreted as a gradual and steady "experience" of the different joints and multiple dimensions of the infinite for the increasingly complex and important opened fields of research.

To use an expression made earlier by Hélène Metzger and followed by Georges Canguilhem, who were historians of science and French epistemologists, the infinite, thanks to its on going 'enjeux' has become an "a priori de l'esprit" leading to get results *a posteriori* which can also be far from its place of origin[12]. This particular capability of a heuristic nature that the infinite has, gives way, within each theory, for better a hold of the intrinsic boundaries and limits, whose critical understanding leads to new conceptualizations, to "progressive conceptualizations" in the sense given by Enriques[13]. The history of XXth century logical-mathematical thinking can be interpreted as a gradual

10 See WEIL 1966 e 1982; on this see CASTELLANA 2004, chap. IV.

11 ENRIQUES 1911, pg. 1-24 now in ENRIQUES 2001.

12 See METZGER 1936, ora in METZGER 2002, pg. 67-78; see GANGUILHEM 1978.

13 See ENRIQUES 1906.

increase in awareness, including the operational implications made in various fields, of the limits of the infinite's static points of view and its close relationship with the important processes of coherence.

Even if only with Cantor has the infinite won its autonomy and a more precise conceptual character, as it is known, it has opened further and more significant cognitive horizons throughout the twentieth century starting off with Hilbert, then with Gentzen and Gödel all the way to the logical-mathematical works of Jean-Yves Girard. It comes as no surprise that the great debates which occurred in the early decades of the 1900's on the foundations of mathematics, were mainly concentrated in trying to clarify the particular nature of the idea of the infinite and in providing a group of proof techniques for the consistency in the corresponding logical theories. The same Gerhard Gentzen in *Der Unendlichkeitsbegriff der Mathematik* (1936-37) proposes a "classification of mathematics" in three different levels according to the role played by the idea of the infinite in diverse theories, from the first level consisting of the elementary theory of numbers , You switch to the second level characterized by the analysis (infinite sets) to reach the third level, thanks to increasingly growing generalizations, represented by the general set theory[14]. As it is known, Gentzen critically analyzes the two now classic interpretations of the infinite's nature in mathematics, the "actualist" (*an sich*) and the "constructivist" (*Konstruktiv*), trying to grasp the differences and convergences within various supporters (Kronecker, Poincaré, Brouwer, Weyl). Although the two interpretations are considered to be "defensible", he sees in Hilbert's proof theory a way to clarify relations between them through "pure mathematical investigations"[15]. The three levels (elementary number theory, analysis and set theory) show the increasing degree of conceptual complexity attained by the idea of the infinite; they require, therefore, particular proof techniques and the rule of transfinite induction becomes a crucial rule in order to determine the soundness in the succession of the consistency proofs.

Gentzen, in his logical-mathematical contributions, thanks to the introduction of the concept of cardinal and to the remarks on the implications of Gödel theorems, turns out to give a concrete shape in a "dynamic" sense of the infinite. That is why, on the one hand, he considers it to be useful for generating finitistic effects and, on the other hand, he frees it from a view based on a re-

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14 GENTZEN 1936-37, pg. 65-80.

15 GENTZEN 1936-37, pg. 76.

ality corresponding to it[16]. The sequent calculus is, therefore, the result of a process based on "dynamic of the proofs" thanks to two different types of operations different between them (construction of statements through rules and proof construction (Schnitt), which is a "composition" operation on the proofs[17]. Such an anti-essentialist view[18] of the infinite, which already Enriques defined as "non-static" due to its constant presence on historical ground[19], is at the bottom of the results made by linear logic, particularly by Jean-Yves Girard.

Many contributions made by the French scholar on logical-mathematical thought are the constant setting for the "enjeux" of the infinite, of its being "mobile", of its "open" character in a Bachelardian sense giving way to new pathways of research . This explains the "turning point", the "Wende" made by Girard in the field of logic in the twentieth century's second half, a turning point that goes by the name of "logical interactionism", still not fully understood in its different aspects[20]. Precisely, in this research area, the constant practice of the notion of the infinite enables him to have a dynamic view of logical thought, where geometric, physical, algebraic concepts, and computer practices 'interact' with each other, and "re"-establish, "re'-organize, and "re"-process the very notion of logical truth[21].

The Girardian works on linear logic and the most recent ones on Ludics, are sort of training establishment where the idea of infinite finally liberates itself from essentialist properties and is able to dynamically analyze all of its joints in depth. Its many "enjeux" explain the leading role that the composition of proofs and the interaction of rules assume in the Girardian works; their dynamic process becomes, as Joinet stated, "l'objet" of logic; so that can you

16 See GENTZEN 1935, pg. 565; on this, see CAVAILLÈS 1938. However, it is good to underline in Hilbert a non static view of the infinite taken as pretence.

17 On this matter see JOINET, *Nature et logique de G. Gentzen à J.Y. Girard,* is still in print.

18 We take this expression from GIRARD 2007 and on the relationship between Gentzen and Girard, see JOINET cit.

19 Cfr. ENRIQUES 1911.

20 Cfr. GIRARD 2006 and 2007.

21 We use these expressions according to the indications of Gaston Bachelard, who speaks about scientific rationalism as a continuous "rationalism of the ri'; cfr. BACHELARD, 1949 and on this CASTELLANA 2004.

can speak of a "refondation girardienne"[22] for the matter respecting the foundations of the logical-mathematical thinking. The linear logic proposes once more the issue of the statute of the infinite, regarded as a 'dynamic infinite' because of the concrete forms in the interaction of rules and due to the multilevel presence inside it[23]. Girard highlights the «règles structurelles génératrices d'infini», the «sources procedurales de l'infini dynamique» well evident in the «traces formelles de l'infini en logique (que sont) les exponentielles»[24]. Each level of the dynamic infinite is characterized as a set of processes that implement various forms of open calculating systems and not closed as "natural" processes that are limited in their ability of representation. Therefore, the infinite, in the work of Girard, is presented under a triple form: dynamic, procedural, and interactive, almost natural in the sense that its "enjeux" are an expression of a "logique de la nature" and fall into "à l'école de la nature"[25]. All this explains girardian criticism to insiemistic belief, to the «tournant formaliste» and the need

> d'un renversement conceptuel: traditionnellement, on de manipulate the syntax (réécriture), ce qui crée une dynamique purement panels. Mon point de vue est plutôt: la dynamique et préexiste the syntax (formules, démonstrations etc.) N'est qu'un commentaire sur des objets almost 'physiques' ...'[26].

With the awareness of the role played by the infinite, Girard, in his works, develops the "natural logic" as already suggested by Gentzen and comes down to outline a conceptual change that invests, primarily, in the epistemic status of logic as a discipline, "la nature de la logique"[27]. The dynamic infinite, therefore, opens the door, to several and new axiomatic reorganizations, to new proof theories, to new systems of transformation for the techniques of demonstration, and introduces in the "sous-sol" of the logic "les principes de

22 JOINET, cit.
23 Cfr. GIRARD 1987 and 1998.
24 GIRARD 2007, p. 397.
25 GIRARD 2007, pg. 407-08.
26 GIRARD 2000, pg. 2-3.
27 JOINET, cit.

l'interaction"[28]. It allows, in fact, to understand the dynamics of logic before its syntax, its truths in motion, and introduces, into the same logic, the principles of transformation, revision, and interaction through re-axiomatizing processes in progress.
In both the linear logic and the Ludics outlined by Girard, the notion of infinite goes back to play a role proportional to its intrinsic problematicity, to be an indispensable tool for the knowledge of reality, once it was freed from the bottlenecks imposed by the "tournant formalist"; as Enriques and Weyl have repeatedly stressed in their writings. Because of the role assigned to the idea of infinite, the Girardian contributions on logical-mathematical contemporary thinking, allow to trace more effective routes, that are aimed to outline various processes for the formalization of the dynamics of knowledge. At the same time, they stress the natural ability of the cognitive processes to self-organize. They also allow an understanding of the logical reasons for these dynamizational processes of knowledge and provide concrete tools for them to operate in varied contexts of research. To quote an expression by Leonardo da Vinci, "lo speculatore delle cose" "the speculator of things]" must aim to seize the "infinite ragioni del reale silente" or "[real reasons for the silent reality]" and the dynamic infinite, which emerges from the Girardian works, gives voice to the naturality of the reasons in knowledge and its special processes, that are turned to fathom the multiple levels of reality.
From a more general epistemological point of view, what has been regarded as the "Girardian logic" made by Girard, thanks to the understanding "des enjeux" of the infinite, gives way for logical-mathematical thought to open up to the reasons of the world. It also gives way to more varied and diverse reasons and ways of knowledge of reality, and to provide more appropriate conceptual and operational tools for the increasingly complex processes of self-organization in knowledge. 'The logical interactionism proposed by Girard allows, then, for logic to open up to the world and its reasons. On the one hand, it can be "open" to the possibilities provided by other areas of investigation receiving encouragement and motives for change in its epistemic statutes; on the other hand, it can give them tools of thought with an increasing conceptual ability of coping with the processes of the dynamics of knowledge[29].

[28] Cfr. GIRARD 2006.

[29] In the last years many meetings, on the wake of Girard's works, especially in France and Italy, (congresses, workshops, summer schools, etc.) are making clear this role of 'overture' of logic and are outlining many directions of the scientific and epistemological research

The linear logic and Ludics proposed by Girard are therefore chapters of logical thinking, developed in the course of the past century and up to now they are promoters for ulterior contributions. Not only have they built- the idea of infinite in the "tissu" of their conceptual articulations, but they also have enriched new and different perspectives making it fruitful in varied research horizons. In order to understand the Girardian "Wende", in all of its complete historical-epistemological thickness, it is necessary to seize "les enjeux" of the infinite and the same founding role of the dynamic infinite, which have given a new driving force and a different light to logic. At the same time, it is vital to critically question another point: how the introduction of the idea of "dynamic infinite" has allowed for logical thinking to achieve "ouvertures" in relation to reality and to the different forms of knowledge modality developed by other grounds of research. Moreover, taking into account its role in current research on logic, it is necessary to start questioning the challenge that the dynamic infinite poses for other research areas, certainly not to dictate alleged scientific models to be followed, but to push them to find new autonomous modalities that are more heuristically fruitful and productive.
At the same time, due to the turning point marked in the most recent logical thinking, the idea of dynamic infinite is a challenge even for epistemological reflection. A challenge that when trying to seize the implicit philosophy and the "pensée des sciences", as was said almost in unison by Federigo Enriques and Gaston Bachelard on one side and Moritz Schlick on the other, is pushed to renew its conceptual apparatus, in order to re-forge its interpretative tools. The critical reflection on Kunhian's "essential tensions" between the old and new in the dynamic of scientific knowledge and the historical-conceptual analysis on the meaning of scientific thinking[30] are, in fact, the basis of an increase in our "epistemological heritage". This increase is renewed precisely thanks to the cognitive increases determined by the development of the "techno-scientific heritage".

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oriented in that way. Cf. The Summer School in Cerisy-La-Salle (19-26 September, 2006), 'Logique et interaction: vers une géométrie de la cognition' ; the Roman meetings at the University 'Roma Tre' (Dicember 2004) on 'La logica lineare e suoi sviluppi'.

30 See F. Enriques of 1934; see ENRIQUES 2007.

Within the cultural limitations of a complex society such as the one we live in, the concept of infinite surpasses the theological horizon in order to place itself in a level of scientific research posing different issues to be analyzed. One of its important applications is in mathematics. If you intended to trace its historical path, you would have to outline the entire Western cultural history, and the nature of this concept would be even more polychrome if you wanted to undertake a journey of the historical reconstruction of its meaning by inserting it into the context of Eastern cultures. Shearing a temporal, and in particular, a cultural demarcation, the role that the notion in question plays in the field of the last century's mathematics' scientific research, can represent the profile from which this contribution intends to focus its attention and reflection on.
In media res, the infinite is one of the issues that gives way to scientific research, particularly, the logical-mathematical type of the twentieth century. David Hilbert, in the famous report he presented in the International Congress at Paris on the year 1900, with all the force of his intellectual authority, places the issue of infinite at the first place among the primary objectives that mathematicians were to meet during the rising century[31].
Many scientific discoveries of the XIXth century had predicted the scenarios that were slowly revealing themselves to the eyes of scientists, and the rise of non-Euclidean geometry had particularly captured their attention. The Theory of Relativity, then, helped to reinforce hopes and disappointments of many, but the main issue that branded the historical path of the use, and for some even an abuse of the infinite in mathematics, were the researches held under the set theory of Georg Cantor. In short, if one imagines, as proposed by Gerhard Genzen[32], to classify mathematics according to the degree of usage of the concept of infinite, set theory would represent, undoubtedly, its peak.
The research around the issues raised by the set theory constitutes the means through which you can rebuild the "psychology" of scientists respecting the infinite; so that it can express itself as a full-blown battle, that is, a cultural one, between two opposing sides. As Henri Poincaré had stressed already in the early twentieth century, the front of the comparison is represented by the following two positions: on one side, there are those scientists who believe that the infinite exists because it is constituted by the infinity of possible things, while on the other side, there are those who believe that the infinite is

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[31] HILBERT, 1902.Topping the list of the problems enunciated by Hilbert is the problem of the continuum.
[32] GENZEN, 1936-37.

"an sich", and that you can obtain the finite after cutting out a small piece from the infinite.

Consider the elementary number theory as a particularly suitable case where the infinite is constituted in its simplest form, that is, as an "infinite sequence of natural numbers". The "actualists" will consider the succession as a complete totality. On the other hand, "constructivists" will consider the sequence data as *in fieri*. The two positions seem insignificant until you consider demonstrations and universal propositions like "all natural numbers that have property P". But the romance between the two perspectives ends from the moment when one considers demonstrations and existential propositions such as "x is a number that has property P". Aristotelically, the categories that filter this concept were and remain "potency" and "act".

These two positions, apparently only an expression of scientific "belief", are reinvigorated and fomented by a variety of corollaries developing full-blown epistemological paths. They are a concrete expression of the implicit Bachelardian epistemology of scientists that is a product of their mathematical view of the world.

Georg Cantor, therefore, inherits an (*a priori*) model of thought for which the actual infinite, if it exists, is unique and absolute and beyond that we cannot go further. However, reflecting on the nature of "numbers", he found that infinities were, in fact, different and therefore, could be ordered. First of all, the father of the classical set theory[33] points out that the numbers (finite or not) can be viewed in two ways: ordinals (1,2,3 ...) and cardinals (an sich). And this possibility of having an order for sets, opened up the matter on the possibility of dealing with different infinities. For these reasons we begin with Cantor to assume that the smallest "transfinite" ordinal, which is obtained when counting gradually all whole numbers, is a full-blown entity, . The important idea of Cantor, who made set theory a new and fruitful field of study, was in stating that two sets A and B have the same number of elements (the same cardinality) if there is a way to fully pair the elements of A with the elements of B. Then the set N of natural numbers has the same cardinality of set Q of rational numbers (both defined as countable), even if N is really a subset of Q. On the other hand, set R of real numbers does not have the same cardinality of N or Q, but a greater one (its defined as not countable). Cantor gave two demonstrations of the uncountablity of R, and in particular the se-

33 CANTOR, 1983.

cond one, which utilizes the diagonal construction, has had an extraordinary influence and countless applications in mathematics and logic.
Cantor went further and built an infinite hierarchy of infinite sets, ordinal and cardinal numbers. This process was controversial in its time, and had received the strong opposition of the finitist Leopold Kronecker, even though nowadays there is no significant disagreement among mathematicians about the fairness on Cantor's ideas. In addition, due to the fact that Cantor introduced the concepts of cardinality and cardinal number to be able to compare transfinite sets between them, he proved the existence of infinite sets with different cardinality, such as natural numbers and real numbers. This opened the door to "the hypothesis of the continuum", that is, using an expression by Cantor himself: the hypothesis where no set exists, whose cardinality is strictly between t ... ed language:

$$\exists A : \aleph_0 < |A| < 2^{\aleph_0}$$

As Enrico Bombieri[34] stated, to consider sets whose elements are at the same time sets, leads almost inevitably, to contradictions. For these reasons the classic set theory was, as is well known even by Cantor, immediately discovered in apodictical contradiction. Bombieri also recalls that when facing these difficulties, the possible answers around which mathematicians were divided into, were three: 1.the one which could be defined as "the axiomatic way" from Zermelo- Fraenkel, 2. The "constructivistic" one from Hilbert and 3. The intuitionistic one from Brouwer.
The axiomatic way spread particularly around research on abstract algebra and participates on the process which, by the second half of the Nineteenth century, tends to the axiomatization of the various fields of mathematics. Precisely this axiomatization process, first used by S then by Zermelo and Fraenkel, allows the revision of the antinomies of the "naive" set theory, and above all, the very definition of a set, in light of a new axiomatic apparatus, defined as follows:

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34 <http://www.math.it/eventi/BOMBIERI.pdf>

Axiom I. *Bestimmtheit* or **Axiom of determination**. If every element of a set M is also an element of N, and vice versa, then M = N, so each set is determined by its elements.

Axiom II. *Elementarmengen* or **Axiom of the elementary sets**. There is a improperly set, the empty set, which contains nothing. If *a* is an object in the domain [for Zermelo, synonymous of sets], there is the set {*a*}, which contains only
element *a*. If *a* and *b* are objects of the domain, then there exists the set { *a*, *b*}, which contains the elements *a*, *b* , and they only.
Axiom III. *Aussonderung* or **Axiom of separation.** If a predicate F (x) is defined for all elements of a set M, then M is a subset MF that contains elements such as all elements x of M for which the predicate F (x) is true.
Axiom IV. *Potenzmenge* or **Axiom of power set.** For each set T there is a set *P* (T) that contains as elements all and only the subsets of T.
Axiom V. *Vereinigung* or **Axiom of union**. For each set T there is a set *S* (T) that contains as elements all and only the subsets of T which contain a single element.
The name of Zermelo is linked moreover to the *axiom of choice*[3535], according to it, given any set of non-empty disjointed sets, it is possible to choose an element, in any set, such that these elements constitute a new set. Its original form is the following:
Axiom VI. *Auswahlt* or **Axiom of choice**; . If T is a set of elements which are not pairwise disjointed empty sets, then the union *S* (T) contains at least a subset *S*1 having one and only one element in common with each element of T.
Axiom VII. *Axiom des Unendlichen* or **Axiom of infinity**. The domain contains at least a set Z containing, at the same time, the null-set as its element, and is constructed so that each element *a* corresponds to an additional {*a*}[3636]

35 RUBIN – RUBIN, 1963; MONK, 1972. pg. 151-165.

36 You can get the infinite set constituted by: 0, {0}, {{0}}, ... This schematic synthesis of the axioms of theory ZF was taken from D'AMORE - MATTEUZZI, 1975, pg. 96.

The epistemological novelty proposed in the present is extrapolated from axiom VI, which is not focusing the attention on the element of the set, but on the relations that it establishes with other elements. It has produced an essentially unstoppable process of development in the direction of what is currently known as group theory.
Returning to the Cantorian contributions, Ettore Casari, when reviewing the concept of set,
rightly notes that:

> More than half a century of arguments and developments of the Cantorian theory allow us to indicate the characteristics of the cantorian concept of sets with the following properties:
> 1) Its *existence* in correspondence to *every multiplicity* of characterizable distinguished entities from a condition.
> 2) Its complete *determination* by the part of the elements of the corresponding multiplicity.
> 3) Its *substantiality* in the two folded aspect of:
> a) *individuality*, and that is, the ability equal to every other individual substance that counts with attributes, in being, that is, an element of multiplicity;
> b) *absoluteness* and that is, an independence from language in the sense that a set and its properties is independent from any linguistic theoretical possibility to characterize them[3737].

It was therefore necessary to proceed to a conceptual correction, that is, a more detailed specification of the concept of set. And this is the main theoric objective that the axiomatic theory of Zermelo - Fraenkel tries to respond to. The axiomatization process focuses on developing a scientific theory by setting a set of propositions that we put on the bottom of the same theory.
However, despite the efforts, says Bombieri

> Gödel and Cohen showed that the axiom of choice C is independent from ZF, so if ZF is a non contradictory

[37] CASARI, 1964, pg. 21.

> theory, then ZF + C is not contradictory; but also ZF + (negation of C) is not contradictory, giving rise to two separate mathematical theories.
>
> Axiom C is very convenient in some respects and is accepted by the majority of mathematicians. Although, not all mathematicians are happy with C, since it leads to results which do not comply with the intuition from the real world.

Therefore, it cannot resolve the matter of the interpretative duality of the continuum. The Hilbertian constructivist approach, on the other hand, reinforces the idea that the infinite is only a Kantian idea, that is, "a concept of reason that goes beyond any experience and through which the concrete is integrated in a sense of totality"[38]. So therefore, mathematics must deal only with finite bodies and their precise rules of deduction. The Berlin master recalls that the paradoxes which the Cantorian theory has led us to, have to be exceeded without betraying science and in order to do that, we must walk the course set by these two basic tracks:

> 1 Wherever there is even a slight hope, we will carefully examine all the conceptualization and fruitful reasoning, consolidating and making them liable to use. From the paradise which Cantor has created for us no one should ever drive away.
>
> 2 It is necessary, to restore, wherever possible, that security in reasoning, which exists in the ordinary inferior number theory, which nobody doubts about and where contradictions and paradoxes arise only for our inattention.
>
> 3 Obviously these objectives can be achieved only if we can give a complete clarification of the nature of the infinite[39].

The Brouwerian intuitionism also bypasses the problem at its core, indicating the limit of the mathematical concerns in the objects that can be defined by a

38 HILBERT, 1925, pg. 189-190.

39 HILBERT, 1925,

few constructive processes of the mind. In fact, all of mathematics is based on the study of entities of which the construction process can be identified[40].

A special place is occupied, in this context, by the epistemological contribution of Hermann Weyl. In several parts of his work, he tackles with the problem of the vicious circle in which the analysis falls into when faced with the attempts to clarify its approach towards the "infinite", the continuum and with the attempts in defining the concept of number.

By holding the reference plane firm along with the sequence of natural numbers as a comparison in order to understand the extent of the problem of the infinite, Hermann Weyl recalls an important fact: that if we should take into account the properties affecting that succession, it will be essential that all the properties (or operations) of numerical objects are introduced by "complete induction". That is to say, if we establish to choose the constructivist path and therefore follow the definition of the infinite *in fieri*, it is necessary for the following to take place: A. the property is valid for 1 and B. the same property is valid for any other number of the succession. In algorithmic form:

$$a+1=a'$$
$$a+n'=(a+n)'$$

One obvious consequence is that the inductive process (complete or mathematical induction) should be valid not only for concepts, properties or operations, but also for proofs. Consequently, the numbers will be distinguished depending on the place they occupy in the succession. From such succession derives the first real epistemological matter for the interpretation of the infinite: given that the numbers can be distinguished depending on the place they hold in the succession, does the primitive concept of number exist and what is it? Cardinal or ordinal? From the above-mentioned, Cantor considered sufficient to assign a number to each set, but Weyl reminds us that "it remains in-

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[40] According to Brouwer, intuitionism is based on two fundamental actions, both a-linguistic and in direct reference to temporal intuition. The first act recognizes that the origin of the mathematical activity derives from the perception of the flow of time, that is, of the division of the immediate unit in two distinct units "one of which gives way to the other, but it is conserved in the memory"; when you foresee the obtained " bi-unit" from any qualitative consideration, it constitutes the pure and empty quantitative form of the number entity. The second act recognizes the possibility to generate successions of free choices proceeding to the infinite, once the terms have been chosen between the already constructed mathematical entities.

dispensable to order each individual set by arranging its elements one by one in a temporal succession"[41].

Hence, in step with the current scientific conceptions, the primitive should be the concept of ordinal number.

Another issue that has animated and still animates the debate on the foundations of mathematics is that of the types of numbers: are numbers ideal entities or symbols? Helmholtz argued that the number theory is nothing more than a "method built upon purely psychological facts"[42]. Hilbert then, developed that idea in a complete and coherent way, leading us to the following consideration: even when you want to see the number as an ideal subject, we must refrain from the designation of an independent existence, and above all, as Weyl still states, "their existence exhausts itself in the developed function and in the reciprocal relations of greater or lesser value"[43].

The last epistemological "enjeux" that the German mathematician poses as a pillar for his reflection on the infinite is the ratio between number, space, and time. In that regard, we can simply recall the work of the great master of Koenigsberg to agree that the link does exist, but Weyl believes that Kantian conclusions are too difficult for twentieth century scientific knowledge. In the light of the axiomatic theory, it is reductive to proclaim that arithmetic is the science of time and geometry is the science of space. Particularly, after having supported the generation of the series through the addition of unity, the staticity of Being is projected on the dynamic background of the possible of a multiplicity produced by an open and unfinished process.

Following these discussions we can agree that the "constructive knowledge" is constituted by these fundamental issues:

> To that which is given, we ascribe certain characters that are not evident in the phenomena, but that we can reach as a result of certain mental operations. It is essential to admit that the performance of these operations is universally possible and that their result is to be uniquely determined by the data. But it is not essential for the operations which define that character to have been carried out.

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41 WEYL, 1949, pg.

42 HELMHOLTZ, 1987, pg. 359

43 WEYL, 1949, pg. 36.

> With the introduction of symbols, the assertions are articulated so that one part of the operations is shifted to the symbols and thereby made independent by what is given and by the fact that it continues to subsist. Therefore, the free manipulation of concepts is contrasted with their application and the ideas become detached from reality and acquire a relative independence.
>
> The characters are not pointed out as they actually occur, but their symbols are projected on the background of an ordered multiplicity of possibilities generated by a fix process and opened towards infinity[44].

Hence, to develop a mathematical theory it is necessary, according to the German mathematician, to have a base category and a primitive relation so that the entire math is based on the ability to decide what is infinite according to its essence on the basis of finite criteria. That way the infinite, with its transcendental nature, will stimulate mathematical knowledge; but we have to settle for the symbol and for all the rules that explain its use without claiming "that the transcendent falls into the enlightened circle of our intuition"[45].

44 WEYL, 1949, pg. 37-38.

45 WEYL, 1949, pg. 66.

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